

# THE IRREDUCIBLE CHARACTERS OF THE ALTERNATING HECKE ALGEBRAS

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**ABSTRACT.** This paper computes the irreducible characters of the alternating Hecke algebras, which are deformations of the group algebras of the alternating groups. More precisely, we compute the values of the irreducible characters of the semisimple alternating Hecke algebras on a set of elements indexed by minimal length conjugacy class representatives and we show that these character values determine the irreducible characters completely. As an application we determine a splitting field for the alternating Hecke algebras in the semisimple case.

## INTRODUCTION

Mitsuhashi [15] introduced a deformation, or  $q$ -analogue, of the group algebras of the alternating groups. He defined these algebras by generators and relations. These algebras can also be realized as the fixed point subalgebra of the Iwahori–Hecke algebra of the corresponding symmetric group. In the semisimple case, Mitsuhashi showed that all of the irreducible representations of these algebras can be obtained by Clifford theory from the irreducible representations of the Hecke algebras of type  $A$ .

The aim of this paper is to explicitly compute the irreducible characters of the alternating Hecke algebras in the semisimple case. As with the alternating groups, most of the irreducible representations of the Iwahori–Hecke algebra of type  $A$  restrict to give irreducible representations of the alternating Hecke algebra. The representations that do not remain irreducible on restriction split as a direct sum of two non-isomorphic irreducible representations so it only these characters that we need to consider. We compute these characters by extending ideas of Headley [8].

One striking feature of the Iwahori–Hecke algebras is that their characters are determined by their values on any set of standard basis elements indexed by a set of minimal length conjugacy class representatives. This was first proved in type  $A$  by Starkey [20] (and later rediscovered by Ram [16]), and then proved for all finite Coxeter groups by Geck and Pfeiffer [3]. In the final section of this paper we show that an analogue the Geck–Pfeiffer theorem holds for the alternating Hecke algebras. Unlike in the Geck–Pfeiffer theorem, the characters of the alternating Hecke algebras are not constant on minimal length conjugacy class representations and, by necessity, our proof relies on a brute force calculation with characters.

The outline of this paper is as follows. In section 1 we define the alternating Hecke algebra and prove some basic results about it. Section 2 constructs the irreducible representations of the alternating Hecke algebras and reduces the calculation of their characters to what is essentially a calculation in the Iwahori–Hecke algebra of the symmetric group. Section 3 computes the irreducible characters for those representations of the Iwahori–Hecke algebras that split on restriction. In section 4 we prove a weak analogue of the Geck–Pfeiffer theorem for the alternating Hecke algebras and, by way of examples, show that a stronger result is not possible for the bases we consider. As an application, we determine a splitting field for the semisimple alternating Hecke algebras.

## 1. THE ALTERNATING HECKE ALGEBRA

In this section we define the alternating Hecke and construct a basis for it.

Through out this section let  $R$  be a commutative ring with one such that 2 is a unit in  $R$ . Let  $q$  be an invertible element of  $R$  such that  $q \neq 1$ .

Fix a positive integer  $n \geq 1$  and let  $\mathfrak{S}_n$  be the symmetric group of degree  $n$ .

**1.1. Definition.** *The Iwahori–Hecke  $R$ -algebra of type  $A$ , with parameter  $q$ , is the unital associative  $R$ -algebra  $\mathcal{H}_q(\mathfrak{S}_n) = \mathcal{H}_{R,q}(\mathfrak{S}_n)$  with generators  $T_1, \dots, T_{n-1}$  and relations*

$$\begin{aligned} (T_i - q)(T_i + q^{-1}) &= 0, & \text{for } 1 \leq i < n, \\ T_j T_i &= T_j T_i, & \text{for } 1 \leq i < j-1 < n-2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, & \text{for } 1 \leq i < n-2. \end{aligned}$$

Define  $s_i = (i, i+1) \in \mathfrak{S}_n$  and  $T_{s_i} = T_i$ , for  $i = 1, \dots, n-1$ . Then  $S = \{s_1, \dots, s_{n-1}\}$  is the standard set of Coxeter generators for  $\mathfrak{S}_n$ . Every element  $w \in \mathfrak{S}_n$  can be written in the form  $w = s_{i_1} \dots s_{i_k}$ , for some  $i_j$  with

$1 \leq i_j \leq k$  for  $j = 1, \dots, k$ . Such an expression is **reduced** if  $k$  is minimal, in which case we write  $\ell(w) = k$  and say that  $w$  has **length**  $k$ .

Suppose that  $x = s_{i_1} \dots s_{i_k}$  is a reduced expression for  $n$ . Set  $T_x = T_{i_1} \dots T_{i_k}$  and  $\ell(x) = k$ . Then, as is well-known,  $\ell: \mathfrak{S}_n \rightarrow \mathbb{N}$  is the usual Coxeter length function on  $x$  and  $T_x$  is independent of the choice of  $i_1, \dots, i_k$ . Moreover,  $\{T_x \mid x \in \mathfrak{S}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Proofs of all of these facts can be found, for example, in [14, Chapter 1].

**1.2. Remark.** We have rescaled the generators of  $\mathcal{H}_q(\mathfrak{S}_n)$  when compared with Iwahori's original definition [11] of  $\mathcal{H}_q(\mathfrak{S}_n)$ . As a result, when comparing our results with the literature it is necessary to replace  $q$  with  $q^{\frac{1}{2}}$  and  $T_w$  with  $q^{-\frac{1}{2}\ell(w)}T_w$ , for  $w \in \mathfrak{S}_n$ . The algebra  $\mathcal{H}_q(\mathfrak{S}_n)$  is cellular by [14, Theorem 3.20], so any field is a splitting field for  $\mathcal{H}_q(\mathfrak{S}_n)$ . Therefore, adjoining a square root of  $q$  to the ground ring does not affect the representation theory of  $\mathcal{H}_q(\mathfrak{S}_n)$ , however, adjoining a square root does affect the representation theory of the alternating Hecke that we introduce below. Later we place further conditions on the ground ring to ensure that we are working over a splitting field for the semisimple alternating Hecke algebras.

By the first relation in Definition 1.1,  $T_s$  is invertible with  $T_s^{-1} = T_s - q + q^{-1}$ , for  $s \in S$ . Consequently,  $T_x$  is invertible, for  $x \in \mathfrak{S}_n$ . For  $x \in \mathfrak{S}_n$  set  $\varepsilon_x = (-1)^{\ell(x)}$ .

### 1.3. Definition.

- a) Let  $\#$  be the unique  $R$ -linear automorphism of  $\mathcal{H}_q(\mathfrak{S}_n)$  such that  $T_x^\# = \varepsilon_x T_{x^{-1}}$ , for all  $x \in \mathfrak{S}_n$ .
- b) The **alternating Hecke algebra**, with parameter  $q$ , is the  $\mathcal{H}_q(\mathfrak{S}_n)$ -fixed point subalgebra

$$\mathcal{H}_q(\mathfrak{A}_n) = \{a \in \mathcal{H}_q(\mathfrak{S}_n) \mid a^\# = a\}$$

Iwahori [11, Theorem 5.4] attributes the involution  $\#$  to Goldman.

The algebra  $\mathcal{H}_q(\mathfrak{A}_n)$  was first considered by Mitsuhashi [15] who defined it by generators and relations. We show in Proposition 2.2 below that Definition 1.3 agrees with Mitsuhashi's definition.

Recall that the **alternating group**  $\mathfrak{A}_n$  is subgroup of  $\mathfrak{S}_n$  consisting of even permutations. Explicitly,  $\mathfrak{A}_n = \{z \in \mathfrak{S}_n \mid \ell(z) \equiv 0 \pmod{2}\}$ . Note that if  $q^2 = 1$  then  $\mathcal{H}_{\pm 1}(\mathfrak{S}_n) \cong R\mathfrak{S}_n$  and  $\mathcal{H}_{\pm 1}(\mathfrak{A}_n) \cong R\mathfrak{A}_n$  since, in this case,  $T_z^\# = \varepsilon_x z$  for all  $z \in \mathfrak{S}_n$ .

Let  $\leq$  be the **Bruhat order** on  $\mathfrak{S}_n$ . Thus, if  $z = s_{i_1} \dots s_{i_k}$  is a reduced expression for  $z$  then  $y \leq z$  if and only if  $y = (j_1, j_1 + 1) \dots (j_l, j_l + 1)$ , where  $j_a = i_{f(a)}$  for some monotonically increasing function  $f: \{1, \dots, l\} \rightarrow \{1, \dots, k\}$ . If  $y \leq z$  and  $y \neq z$  we write  $y < z$  or  $z > y$ . Write  $x \prec y$  if  $x < y$  and  $\ell(y) \not\equiv \ell(x) \pmod{2}$  and  $x \preceq y$  if  $x \prec y$  or  $x = y$ .

We start by describing a new basis for  $\mathcal{H}_q(\mathfrak{S}_n)$  that is compatible with the alternating Hecke algebra  $\mathcal{H}_q(\mathfrak{A}_n)$ . For  $z \in \mathfrak{S}_n$  set

$$(1.4) \quad A_z = \frac{1}{2}(T_z + \varepsilon_x T_z^\#).$$

For example, if  $s \in S$  then  $A_s = T_s - \frac{1}{2}(q - q^{-1})$ .

Set  $\mathcal{H}_q^\pm = \{h \in \mathcal{H}_q(\mathfrak{S}_n) \mid h^\# = \pm h\}$ . By definition  $\mathcal{H}_q^+$  and  $\mathcal{H}_q^-$  are  $R$ -submodules of  $\mathcal{H}_q(\mathfrak{S}_n)$ , however, note that  $\mathcal{H}_q(\mathfrak{A}_n) = \mathcal{H}_q(\mathfrak{S}_n)^+$  is a subalgebra. Abusing notation, write  $\mathcal{H}_q^{\varepsilon_z} = \mathcal{H}_q^\pm$  for the corresponding choice of sign.

**1.5. Lemma.** Suppose that  $z \in \mathfrak{S}_n$ . Then  $A_z = T_z + \sum_{y < z} a_{yz} T_y$ , for some  $a_{yz} \in R$ . Moreover,  $A_z \in \mathcal{H}_q^{\varepsilon_z}$ .

*Proof.* By definition,  $A_z^\# = \varepsilon_x A_z$ , so  $A_z \in \mathcal{H}_q^{\varepsilon_z}$ . To complete the proof observe that if  $z = s_{i_1} \dots s_{i_k}$  is a reduced expression for  $z$  then, as is well-known,

$$T_z^\# = (-T_{s_1} + q - q^{-1}) \dots (-T_{s_k} + q - q^{-1}) = \varepsilon_z T_z + \sum_{y < z} r_{yz} T_y,$$

for some  $r_{yz} \in R$ . Hence, the second statement follows by setting  $a_{yz} = \frac{1}{2}\varepsilon_z r_{yz}$ .  $\square$

**1.6. Proposition.** Suppose that  $n \geq 1$ . Then the following hold:

- a)  $\{A_z \mid z \in \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{A}_n) = \mathcal{H}_q^+$ .
- b)  $\{A_z \mid z \in \mathfrak{S}_n \setminus \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q^-$ .
- c) As an  $R$ -module,  $\mathcal{H}_q(\mathfrak{S}_n) = \mathcal{H}_q^+ \oplus \mathcal{H}_q^-$ .

*Proof.* By Lemma 1.5, the transition matrix between the  $T$ -basis  $\{T_z \mid z \in \mathfrak{S}_n\}$  and the basis  $\{A_z \mid z \in \mathfrak{S}_n\}$  is unitriangular, so  $\{A_z \mid z \in \mathfrak{S}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Applying Lemma 1.5 again,  $A_z \in \mathcal{H}_q^{\varepsilon_z}$ . Hence, all parts of the proposition now follow since  $\mathcal{H}_q^+ \cap \mathcal{H}_q^- = 0$ .  $\square$

In particular,  $\{A_w \mid w \in \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{A}_n)$ . When  $q^2 = 1$  the basis element  $A_w$ , for  $w \in \mathfrak{A}_n$  coincides with the basis element  $w$  in the group ring  $R\mathfrak{A}_n$ , however,  $A_w$  typically involves many terms  $T_y$  with  $y \leq w$  and  $y \notin \mathfrak{A}_n$ . The algebra  $\mathcal{H}_q(\mathfrak{A}_n)$  has another basis  $\{B_z \mid z \in \mathfrak{A}_n\}$  that also coincides with the group-basis of  $R\mathfrak{A}_n$  when  $q^2 = 1$  and which does not involve any terms  $T_y$  with  $y < w$  and  $y \in \mathfrak{A}_n$ .

**1.7. Proposition.** a) For each  $z \in \mathfrak{S}_n$  there exists a unique element  $B_z \in \mathcal{H}_q^{\varepsilon_z}$  such that

$$B_z = T_z + \sum_{y \prec z} b_{yz} T_y,$$

for some  $b_{yz} \in R$ .

b)  $\{B_z \mid z \in \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{A}_n) = \mathcal{H}_q^+$ .

c)  $\{B_z \mid z \in \mathfrak{S}_n \setminus \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q^-$ .

*Proof.* Parts (b) and (c) follow directly from part (a) and [Proposition 1.6](#), so it remains to prove part (a). First suppose that there exist two elements  $B_z$  and  $B'_z$  in  $\mathcal{H}_q^{\varepsilon_z}$  that are of the required form. By construction,  $B_z - B'_z$  is a  $R$ -linear combination of terms  $T_y$  with  $y < z$  and  $\varepsilon_y = -\varepsilon_z$ . On the other hand,  $B_z - B'_z \in \mathcal{H}_q^{\varepsilon_z}$  so that  $B_z - B'_z$  is a  $R$ -linear combination of terms  $A_y$ , with  $\varepsilon_y = -\varepsilon_z$ . The only way that both of these constraints are possible is if  $B_z - B'_z = 0$ . Therefore,  $B_z$  is uniquely determined as claimed.

To prove existence we argue by induction on  $\ell(z)$ . If  $\ell(z) \leq 1$  then  $B_z = A_z$  so there is nothing to prove. If  $\ell(z) > 1$  then the element

$$B_z = A_z - \sum_{\substack{y < z \\ \ell(y) - \ell(z) \in 2\mathbb{Z}}} a_{yz} B_y$$

has the required properties, where the coefficients  $a_{yz} \in R$  are given by [Lemma 1.5](#). □

Set  $b_{zz} = 1$ . The next result describes how  $B_r$  acts on the  $B$ -basis.

**1.8. Corollary.** Suppose that  $z \in \mathfrak{S}_n$  and  $1 \leq r < n$ . Then

$$B_r B_z = \delta_{rz > z} B_{rz} + \frac{1}{2}(q - q^{-1}) \left( \sum_{\substack{y \prec z \\ ry < y}} b_{yz} B_y - \sum_{\substack{y \prec z \\ ry > y}} b_{yz} B_y \right),$$

where  $\delta_{rz > z} = 1$  if  $rz > z$  and  $\delta_{rz > z} = 0$  if  $rz < z$ .

*Proof.* Since  $B_r = A_r = T_r - \frac{1}{2}(q - q^{-1})$ , we compute

$$\begin{aligned} B_r B_z &= (T_r - \frac{1}{2}(q - q^{-1})) \sum_{y \preceq z} b_{yz} T_y \\ &= \sum_{\substack{y \preceq z \\ ry < y}} b_{yz} ((q - q^{-1})T_y + T_{ry}) + \sum_{\substack{y \preceq z \\ ry > y}} b_{yz} T_{ry} - \frac{1}{2}(q - q^{-1}) \sum_{y \preceq z} b_{yz} T_y. \end{aligned}$$

If  $B_z \in \mathcal{H}_q^{\pm}$  then  $B_r B_z \in \mathcal{H}_q^{\mp}$  so that  $B_r B_z = \sum_y c_y B_y$ , where  $c_y \neq 0$  only if  $\ell(y) \not\equiv \ell(z) \pmod{2}$ . By [Proposition 1.7](#), if  $\ell(y) \not\equiv \ell(z) \pmod{2}$  then  $c_y$  is equal to the coefficient of  $T_y$  in the last displayed equation, so the result follows. □

**1.9. Remark.** The argument of [Corollary 1.8](#) shows that if  $rz < z$  then  $b_{rz,z} = \frac{1}{2}(q - q^{-1})$ .

The basis  $\{B_z \mid z \in \mathfrak{A}_n\}$  of  $\mathcal{H}_q(\mathfrak{A}_n)$  was independently discovered by Shoji [19] using a construction that is reminiscent of that for the Kazhdan-Lusztig basis of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Shoji's argument gives more information about the coefficients  $b_{yz}$ , for  $z, y \in \mathfrak{S}_n$ . Shoji also showed that each  $B_z$  is invariant under Kazhdan and Lusztig's bar involution [13]. This fact will be useful for us below, so we now give a proof of this.

Assume for the next results that  $R = \mathcal{Z}$ , where  $\mathcal{Z} = \mathbb{Z}[\frac{1}{2}, q, q^{-1}]$  and  $q$  is an indeterminate over  $\mathbb{Z}$ . The bar involution  $\bar{\cdot} : \mathcal{Z} \rightarrow \mathcal{Z}$  is the unique ring involution of  $\mathcal{Z}$  such that  $\bar{q} = q^{-1}$ . A **semilinear** involution on  $\mathcal{H}_q(\mathfrak{S}_n)$  is a  $\mathbb{Z}$ -linear ring involution  $\iota : \mathcal{H}_q(\mathfrak{S}_n) \rightarrow \mathcal{H}_q(\mathfrak{S}_n)$  such that  $\iota(q) = q^{-1}$ . Inspecting [Definition 1.1](#), there are two semilinear involutions of the Hecke algebra  $\mathcal{H}_q(\mathfrak{S}_n)$  given by

$$\beta(T_z) = T_{z^{-1}}^{-1} \quad \text{and} \quad \epsilon(T_z) = \varepsilon_z T_z,$$

for all  $z \in \mathfrak{S}_n$ . (The involution  $\beta$  is the Kazhdan-Lusztig bar involution.)

**1.10. Corollary** (Shoji [19]). Suppose that  $z \in \mathfrak{S}_n$ . Then  $\epsilon(B_z) = \varepsilon_z B_z$  and  $\beta(B_z) = B_z$ .

*Proof.* By Proposition 1.7,  $\epsilon(B_z) = \varepsilon_z T_z + \sum_{y < z} \varepsilon_y \overline{b_{yz}} T_y$ , where  $b_{yz} \neq 0$  only if  $\varepsilon_y = -\varepsilon_z$ . Therefore,  $\epsilon(B_z) = \varepsilon_z B_z$  by the uniqueness clause of Proposition 1.7 (consequently,  $\overline{b_{yz}} = \varepsilon_z \varepsilon_y b_{yz} = -b_{yz}$ ). By comparing the effect of these involutions on the generators of  $\mathcal{H}_q(\mathfrak{S}_n)$  shows that these two involutions commute and that  $\beta = \epsilon \circ \#$ . Therefore,  $\beta(B_z) = \epsilon(B_z^\#) = \varepsilon_x \epsilon(B_x) = B_z$  as required.  $\square$

## 2. MITSUHASHI'S GENERATORS AND RELATIONS

Mitsuhashi [15] defined an alternating Hecke algebra using generators and relations. In this section we show that Mitsuhashi's algebra is isomorphic to  $\mathcal{H}_q(\mathfrak{A}_n)$ . We assume that  $R$  is a commutative ring with 1 such that 2 and  $q + q^{-1}$  are both invertible in  $R$ .

Following Mitsuhashi [15, Definition 4.1], but taking Remark 1.2 into account, define the **Mitsuhashi alternating Hecke algebra**  $\mathcal{A}_q(n)$  to be the unital associative  $R$ -algebra with generators  $A_1, \dots, A_{n-1}$  and relations

$$A_1 = 1, \quad (A_i A_j)^2 = 1 \quad \text{and} \quad \left( (A_{i-1} A_i)^2 + \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 (A_{i-1} A_i - 1) \right) A_{i-1} A_i = 1,$$

where  $2 \leq i, j < n$  and  $|i - j| \neq 1$ . (This is a streamlined version of Mitsuhashi's presentation.)

Mitsuhashi obtained this presentation by first giving a new presentation for  $\mathcal{H}_q(\mathfrak{S}_n)$ . For  $1 \leq i < n$  define

$$E_i = \frac{(2T_i - q + q^{-1})}{(q + q^{-1})}.$$

We record some elementary properties of these elements (except for the first claim, these can be found in [15]).

**2.1. Lemma.** *Suppose that  $1 \leq i < n$ . Then  $E_i^\# = -E_i$ ,  $E_i^2 = 1$  and  $T_i = \frac{(q+q^{-1})}{2} E_i + \frac{1}{2}(q - q^{-1})$ .*

In particular, since  $T_i = \frac{1}{2}((q + q^{-1})E_i + q - q^{-1})$ , it follows that  $\mathcal{H}_q(\mathfrak{S}_n)$  is generated as an  $R$ -algebra by the elements  $\{E_i \mid 1 \leq i < n\}$ .

**2.2. Proposition.** *Suppose that 2 and  $q + q^{-1}$  are invertible in  $R$ . Then  $\mathcal{H}_q(\mathfrak{A}_n) \cong \mathcal{A}_q(n)$  as  $R$ -algebras.*

*Proof.* By Lemma 2.1, the set  $\{E_1 E_i \mid 2 \leq i < n\}$  generates  $\mathcal{H}_q(\mathfrak{A}_n)$  as an  $R$ -algebra since  $\{E_i \mid 1 \leq i < n\}$  generates  $\mathcal{H}_q(\mathfrak{S}_n)$ . As noted in [15, Lemma 4.2] it follows easily from the relations in  $\mathcal{H}_q(\mathfrak{S}_n)$  that  $E_i E_j = E_j E_i$  if  $|i - j| \neq 1$  and that

$$\left( (E_{i-1} E_i)^2 + \left( \frac{q - q^{-1}}{q + q^{-1}} \right)^2 (E_{i-1} E_i - 1) \right) E_{i-1} E_i = 1$$

for  $2 < i < n$ . Hence, there is a unique algebra homomorphism  $\theta : \mathcal{A}_q(n) \rightarrow \mathcal{H}_q(\mathfrak{A}_n)$  such that  $\theta(A_i) = E_1 E_i$ , for  $1 \leq i < n$ . By [15, Theorem 4.7],  $\mathcal{A}_q(n)$  is free as an  $R$ -module of rank  $\frac{1}{2}n!$ , so  $\theta$  is an isomorphism by Proposition 1.6.  $\square$

As a consequence of the proof of Proposition 2.2, if 2 and  $q + q^{-1}$  are both invertible then the  $R$ -module decomposition of  $\mathcal{H}_q(\mathfrak{S}_n)$  from Proposition 1.6(c) becomes an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module decomposition.

**2.3. Corollary.** *Suppose that 2 and  $q + q^{-1}$  are invertible in  $R$ . Then, as an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module*

$$\mathcal{H}_q(\mathfrak{S}_n) = \mathcal{H}_q(\mathfrak{A}_n) \oplus E_1 \mathcal{H}_q(\mathfrak{A}_n).$$

*In particular,  $\mathcal{H}_q(\mathfrak{S}_n)$  is free as an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module.*

Let  $M$  be an  $\mathcal{H}_q(\mathfrak{S}_n)$ -module. Define  $M^\# = \{m' \mid m \in M\}$  to be the  $\mathcal{H}_q(\mathfrak{S}_n)$ -module that is isomorphic to  $M$  as a vector space but where the  $\mathcal{H}_q(\mathfrak{S}_n)$ -action is twisted by  $\#$ . More explicitly, if  $m \in M$  and  $h \in \mathcal{H}_q(\mathfrak{S}_n)$  then  $m'h = (mh^\#)'$ . Observe that  $M \cong M^\#$  as  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules since  $a^\# = a$ , for all  $a \in \mathcal{H}_q(\mathfrak{A}_n)$ .

By Corollary 2.3 and a standard application of Clifford theory (see, for example, [17, appendix]), we have:

**2.4. Proposition.** *Suppose that  $\mathbb{F}$  is an algebraically closed field in which both 2 and  $q + q^{-1}$  are invertible. Let  $M$  be an irreducible  $\mathcal{H}_q(\mathfrak{S}_n)$ -module. Then  $M$  is irreducible as an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module if and only if  $M \not\cong M^\#$  as  $\mathcal{H}_q(\mathfrak{S}_n)$ -modules. Moreover,  $M \cong M^\#$  as  $\mathcal{H}_q(\mathfrak{S}_n)$ -modules if and only if  $M \cong M^+ \oplus M^-$  as  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules, where  $M^+$  and  $M^-$  are non-isomorphic irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules.*

*Proof.* In the notation of [17, Appendix], take  $A = \mathcal{H}_q(\mathfrak{A}_n)$ . Then  $\mathcal{H}_q(\mathfrak{S}_n) \cong \mathcal{H}_q(\mathfrak{A}_n) \rtimes \mathbb{Z}/2\mathbb{Z}$ . The condition that  $M \cong M^\#$  is equivalent to saying that the inertia group of  $M$  is equal to  $\mathbb{Z}/2\mathbb{Z}$ , so the Proposition is a special case of [17, Theorem A.6].  $\square$

## 3. CONJUGACY CLASS REPRESENTATIVES

Before constructing the irreducible representations of  $\mathcal{H}_q(\mathfrak{A}_n)$  we recall the labeling of the conjugacy classes of  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$ . The full details can be found, for example, in [12, Chapter 1].

A **composition** of  $n$  is a sequence  $\mu = (\mu_1, \dots, \mu_k)$  of positive integers that sum to  $n$ . (For convenience, we set  $\mu_0 = 0$  in many places below.) A composition  $\mu$  is a **partition** if its parts  $\mu_1, \dots, \mu_k$  are in weakly decreasing order. The **length** of a partition or composition is  $\ell(\mu) = k$  if  $\mu$  has  $k$  parts. Let  $\mathcal{P}_n$  be the set of partitions of  $n$ .

We identify a partition  $\mu$  with its **diagram**:

$$[\mu] = \{(r, c) \mid 1 \leq c \leq \mu_r \text{ for } r \geq 1\}.$$

In this way, we talk about the rows and columns of (the diagram of)  $\lambda$ . If  $\lambda$  and  $\mu$  are partitions such that  $\lambda_i \leq \mu_i$ , for  $i \geq 1$ , then  $\lambda$  is a **contained** in  $\mu$ ,  $\mu/\lambda$  is a **skew partition**, and we write  $\lambda \subseteq \mu$  and  $\lambda \subset \mu$  if  $\lambda \neq \mu$ . If  $\lambda \subseteq \mu$  then the set

$$[\mu/\lambda] = [\mu]/[\lambda] = \{x \in [\mu] \mid x \notin [\lambda]\}$$

is the **skew diagram** of shape  $\mu/\lambda$ .

If  $\lambda$  is a partition then its **conjugate** is the partition  $\lambda' = (\lambda'_1, \lambda'_2, \dots)$  where  $\lambda'_i$  is the number of entries in column  $i$  of  $[\lambda]$ . A partition  $\lambda$  is **self-conjugate**, or **symmetric**, if  $\lambda = \lambda'$ .

As is well-known, and is straightforward to prove, two elements of the symmetric group are conjugate if and only if they have the same cycle type. Consequently, the conjugacy classes of  $\mathfrak{S}_n$  are indexed by the partitions of  $n$ . If  $\kappa = (\kappa_1, \dots, \kappa_k)$  is a composition of  $n$  let  $C_\kappa$  be the elements of  $\mathfrak{S}_n$  of cycle type  $\kappa$ . Then  $\mathfrak{S}_n = \coprod_{\kappa \in \mathcal{P}_n} C_\kappa$  is the decomposition of  $\mathfrak{S}_n$  into the disjoint union of its conjugacy classes. If  $\kappa = (\kappa_1, \dots, \kappa_d)$  let

$$w_\kappa = (1, 2, \dots, \kappa_1)(\kappa_1 + 1, \kappa_1 + 2, \dots, \kappa_1 + \kappa_2) \dots (\kappa_1 + \dots + \kappa_{d-1} + 1, \dots, \kappa_1 + \dots + \kappa_d).$$

Then  $w_\kappa$  is an element of minimal length in  $C_\kappa$ . In particular,  $\{w_\kappa \mid \kappa \in \mathcal{P}_n\}$  is a complete set of conjugacy class representatives for  $\mathfrak{S}_n$ .

Now consider the alternating group  $\mathfrak{A}_n$ . If  $\kappa = (\kappa_1, \dots, \kappa_k)$  is a partition of  $n$  then  $w_\kappa \in \mathfrak{A}_n$  if and only if  $\ell(w_\kappa) = \sum_{i=1}^l (\kappa_i - 1) = n - \ell(\kappa)$  is even. Equivalently,  $w_\kappa \in \mathfrak{A}_n$  if  $\kappa$  has an even number of non-zero even parts. Let  $\mathcal{P}_n^{\text{alt}} = \{\kappa \in \mathcal{P}_n \mid n \equiv \ell(\kappa) \pmod{2}\}$  be this set of partitions so that  $w_\kappa \in \mathfrak{A}_n$  if and only if  $\kappa \in \mathcal{P}_n^{\text{alt}}$ .

Fix  $\kappa \in \mathcal{P}_n^{\text{alt}}$ . If  $\kappa$  contains a repeated part, or a non-zero even part, then  $w_\kappa$  commutes with an element of odd order, which implies that  $C_\kappa$  is a single conjugacy class of  $\mathfrak{A}_n$ . As a notational sleight of hand, set  $w_\kappa^+ = w_\kappa = w_\kappa^-$  in this case. On the other hand, if  $n > 1$  and all of the parts of  $\kappa \in \mathcal{P}_n^{\text{alt}}$  are all odd and distinct then  $C_\kappa$  is the disjoint union of two conjugacy classes in  $\mathfrak{A}_n$ . Set  $w_\kappa^+ = w_\kappa$  and  $w_\kappa^- = s_r w_\kappa s_r$ , where  $r = \kappa_1 + \dots + \kappa_{d-1} + 1$  and  $d$  is minimal such that  $\kappa_d > 1$ . Then  $w_\kappa^- \in \mathfrak{A}_n$  and  $\ell(w_\kappa^-) = \ell(w_\kappa)$  with  $w_\kappa^\pm$  being conjugate in  $\mathfrak{S}_n$  but *not* conjugate in  $\mathfrak{A}_n$ . In view of [12, Lemma 1.2.10] and [4, Example 3.1.16],  $\{w_\kappa^\pm \mid \kappa \in \mathcal{P}_n^{\text{alt}}\}$  is a complete set of minimal length conjugacy class representatives for  $\mathfrak{A}_n$ , for  $n > 1$ .

The description of the conjugacy classes of  $\mathfrak{A}_n$  is not quite what we expect because, in the semisimple case, it is well-known that the irreducible representations of  $\mathfrak{S}_n$  that are indexed by the self-conjugate partitions split on restriction to  $\mathfrak{A}_n$  (see Proposition 4.10). The combinatorial connection between the representations and the conjugacy classes that split is given by the following definition.

**3.1. Definition.** Suppose that  $\lambda = \lambda'$ . Let  $h(\lambda) = (h_1, h_2, \dots, h_{d(\lambda)})$ , where  $d(\lambda) = \max\{i \mid \lambda_i \geq i\}$  is the length of the diagonal in  $[\lambda]$  and  $h_i = \lambda_i + \lambda'_i - 2i + 1$ , for  $1 \leq i \leq d(\lambda)$ .

By definition, all of the parts of  $h(\lambda)$  are odd and distinct. Pictorially,  $h_i$  is the length of the  $i^{\text{th}}$  diagonal hook  $H_i(\lambda) = \{(i, j), (j, i) \mid i \leq j \leq \lambda_i\}$  in the diagram of  $\lambda$ . For example, if  $\lambda = (k+1, 1^k)$  is a hook partition then  $h(\lambda) = (2k+1)$  and  $d(\lambda) = 1$ .

4. THE IRREDUCIBLE REPRESENTATIONS OF  $\mathcal{H}_q(\mathfrak{A}_n)$ 

This section gives an explicit description of the irreducible representations of  $\mathcal{H}_q(\mathfrak{A}_n)$  when  $\mathcal{H}_q(\mathfrak{A}_n)$  is semisimple. These representations have already been described by Mitsuhashi [15], however, the construction that we give of the irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules is different to his. In the next section we use these modules to compute the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$ .

As we are interested in computing characters we want to work over a field that is large enough to ensure that  $\mathcal{H}_q(\mathfrak{A}_n)$  is a split semisimple algebra. As we will see, this requires that our ground field contains certain square roots. To describe these, for any integer  $k$  define the  $q$ -integer  $[k] = [k]_q$  by

$$[k] = \begin{cases} q + q^3 + \dots + q^{2k-1}, & \text{if } k \geq 0, \\ -q - q^{-3} - \dots - q^{-2k+1}, & \text{if } k < 0. \end{cases}$$



In particular,  $[0] = 0$ ,  $[1] = q$  and  $[-1] = -q^{-1}$ . If  $q^2 \neq 1$  then  $[k] = (q^{2k} - 1)/(q - q^{-1})$ , whereas if  $q^2 = 1$  then  $[k] = k$  for all  $k \in \mathbb{Z}$ . By definition, if  $k > 0$  then  $[-k] = -q^{-2k}[k]$ , for  $k \in \mathbb{Z}$ .

**4.1. Definition.** Suppose that  $\mathbb{F}$  is a field and that  $q \in \mathbb{F}$ . We assume that the elements  $-1$ ,  $[k]$ , for  $1 \leq k \leq n$ , are non-zero and have square-roots in  $\mathbb{F}$ .

Fix a choice of square roots  $\sqrt{-1}$  and  $\sqrt{[k]}$  in  $\mathbb{F}$ , for  $1 \leq k \leq n$ , and set  $\sqrt{[-k]} = q^{-k}\sqrt{-1}\sqrt{[k]} \in \mathbb{F}$ .

For example, if  $F$  is any field and  $q$  is in indeterminate over  $F$  then we could take  $\mathbb{F} = F(\sqrt{-1}, \sqrt{[1]}, \dots, \sqrt{[n]})$ , which is a subfield of  $F((q))$ .

For the remainder of this paper we work over the field  $\mathbb{F}$  and consider the  $\mathbb{F}$ -algebras  $\mathcal{H}_q(\mathfrak{S}_n) = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$  and  $\mathcal{H}_q(\mathfrak{A}_n) = \mathcal{H}_{\mathbb{F},q}(\mathfrak{A}_n)$ . Definition 4.1 implies that  $[1][2] \dots [n] \neq 0$  in  $\mathbb{F}$ . By results going back to Hoefsmit [9], this condition implies that  $\mathcal{H}_q(\mathfrak{S}_n)$  is split semisimple. In fact, since  $\mathcal{H}_q(\mathfrak{S}_n)$  is a cellular algebra [14], any field is a splitting field for  $\mathcal{H}_q(\mathfrak{S}_n)$ . By Corollary 4.11 below,  $\mathbb{F}$  is a splitting field for  $\mathcal{H}_q(\mathfrak{A}_n)$ .

If  $\lambda$  is a partition (or a skew partition) then a  $\lambda$ -**tableau** is an injective map  $t: [\lambda] \rightarrow \mathbb{Z}$ . We think of a  $\lambda$ -tableau as a labeling of the diagram of  $\lambda$  and we say that  $i$  appears in row  $r$  and column  $c$  of  $t$  if  $t(r, c) = i$ .

A **standard  $\lambda$ -tableau** is a bijection  $t: [\lambda] \rightarrow \{1, \dots, n\}$  such that the entries of this tableau increase along rows and down columns. More precisely,  $t$  is standard if  $t(r, c) < t(s, d)$  whenever  $(r, c)$  and  $(s, d)$  are distinct elements of  $[\lambda]$  with  $r \leq s$  and  $c \leq d$ . Let  $\text{Std}(\lambda)$  be the set of standard  $\lambda$ -tableaux and let  $\text{Std}(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda)$ . If  $t \in \text{Std}(\lambda)$  then the **conjugate tableau**  $t'$  is the standard  $\lambda'$ -tableau  $t'$  such that  $t'(c, r) = t(r, c)$ , for  $(c, r) \in [\lambda']$ .

Let  $t$  be a  $\lambda$ -tableau and  $i$  an integer with  $1 \leq i \leq n$ . Then  $t(r, c) = i$  for some  $(r, c) \in [\lambda]$ . The **content** of  $i$  in  $t$  is the integer  $c_t(i) = c - r$ . Observe that if  $t$  is standard then  $c_t(i) > 0$  if  $i + 1$  appears in a later column of  $t$  than  $i$  and, otherwise,  $c_t(i) < 0$ . If  $i < n$  then the **axial distance** from  $i$  to  $i + 1$  is  $\rho_t(i) = c_t(i) - c_t(i + 1)$ .

To compute the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  we need an explicit description of its representations. To this end, for  $k \in \mathbb{Z}$  set

$$\alpha_k = \begin{cases} \frac{\sqrt{-1}\sqrt{[k+1]}\sqrt{[k-1]}}{[k]}, & \text{if } k > 0, \\ -\alpha_{-k}, & \text{if } k < 0. \end{cases}$$

If  $t$  is a standard  $\lambda$ -tableau and  $1 \leq i < n$  then set

$$\alpha_t(i) = \begin{cases} \alpha_{\rho_t(i)}, & \text{if } ts_i \in \text{Std}(\lambda), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if  $t$  is standard then  $\alpha_{t'}(i) = -\alpha_t(i)$ , for  $1 \leq i < n$ . This property of the  $\alpha$ -coefficients is crucial below because it ensures that the Specht modules for  $\mathcal{H}_q(\mathfrak{S}_n)$  that are indexed by self-conjugate partitions split when they are restricted to  $\mathcal{H}_q(\mathfrak{S}_n)$ .

**4.2. Proposition.** Suppose that  $\lambda$  is a partition and let  $S(\lambda)$  be the  $\mathbb{F}$ -vector space with basis  $\{v_t \mid t \in \text{Std}(\lambda)\}$ . Then  $S(\lambda)$  becomes an irreducible  $\mathcal{H}_q(\mathfrak{S}_n)$ -module with  $\mathcal{H}_q(\mathfrak{S}_n)$ -action for  $i = 2, \dots, n - 1$  given by

$$v_t T_i = \frac{-1}{[\rho_t(i)]} v_t + \alpha_t(i) v_{ts_i}.$$

Note that if  $\rho_t(i) = \pm 1$  then  $ts_i$  is not standard and  $v_t T_i = \frac{-1}{[\pm 1]} v_t = \mp q^{\mp 1} v_t$  since  $\alpha_t(i) = 0$ .

*Proof.* Extending the classical arguments for the symmetric group, Hoefsmit [9] has the first to give a seminormal form for the irreducible representations of  $\mathcal{H}_q(\mathfrak{S}_n)$ . This result can be proved by essentially repeating Hoefsmit's argument. Alternatively, it is straightforward to check that if  $t$  and  $v = ts_i$  are both standard tableaux then

$$(4.3) \quad \alpha_t(i) \alpha_v(i) = \frac{[1 + \rho_t(i)][1 + \rho_v(i)]}{q^2 [\rho_t(i)][\rho_v(i)]}$$

since  $\rho_v(i) = -\rho_t(i)$ . Moreover, if  $t$  is standard and  $1 \leq i < n - 1$  then

$$\alpha_t(i) = \alpha_{ts_{i+1}s_i}(i + 1), \quad \alpha_{ts_i}(i + 1) = \alpha_{ts_{i+1}}(i), \quad \text{and} \quad \alpha_{ts_{i+1}s_{i+1}}(i) = \alpha_t(i + 1),$$

so that  $\alpha_t(i) \alpha_{ts_i}(i + 1) \alpha_{ts_{i+1}s_{i+1}}(i) = \alpha_t(i + 1) \alpha_{ts_{i+1}}(i) \alpha_{ts_{i+1}s_i}(i + 1)$ . Similarly, if  $|i - j| > 1$  then  $\alpha_t(i) \alpha_{ts_i}(j) = \alpha_t(j) \alpha_{ts_j}(i)$ . Therefore, the set of scalars  $\{\alpha_t(i) \mid 1 \leq i < n \text{ and } t \in \text{Std}(\mathcal{P}_n)\}$  is a *\*-seminormal coefficient system* for  $\mathcal{H}_q(\mathfrak{S}_n)$  in the sense of [10, §3]. (When comparing the results of this paper with [10], recall Remark 1.2, and note that (4.3) corresponds to the quadratic relation  $(T_i - q)(T_i + q^{-1}) = 0$ . This accounts for the additional factor of  $q^{-2}$  in (4.3) when compared with the corresponding formula in [10].) Hence, the proposition is a special case of [10, Theorem 3.13].  $\square$

We want to describe the Specht modules as  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules. Let  $\tau : S(\lambda) \rightarrow S(\lambda')$  be the  $\mathbb{F}$ -linear map given by  $v_t \mapsto v_{t'}$ , for all  $t \in \text{Std}(\lambda)$ . The map  $\tau$  depends on the partition  $\lambda$ , however, its meaning should always be clear from context.

**4.4. Proposition.** *Suppose that  $w \in \mathfrak{S}_n$ . Then  $v_t T_w^\# = \tau(v_{t'} T_w)$ , for all  $t \in \text{Std}(\lambda)$ .*

*Proof.* Fix  $t \in \text{Std}(\lambda)$ . Suppose first that  $\ell(w) = 1$ , so that  $w = s_i$  for some  $i$ . Let  $\rho_t(i) = c_t(i+1) - c_t(i)$  be the axial distance from  $i$  to  $i+1$  in  $t$ . If  $|\rho_t(i)| = 1$  then  $i$  and  $i+1$  are in the same row or in the same column of  $t$ , and the Lemma follows directly from the definitions. Suppose now that  $|\rho_t(i)| > 1$ . Recall that  $\alpha_{-k} = -\alpha_k$  if  $k \neq 0$ . Therefore,

$$\begin{aligned} v_t T_i^\# &= v_t(-T_i + q - q^{-1}) = \left( \frac{1}{[\rho_t(i)]} + q - q^{-1} \right) v_t - \alpha_t(i) v_{ts_i} \\ &= \frac{-1}{[-\rho_t(i)]} v_t + \alpha_{t'(i)} v_{ts_i} = \tau(v_{t'} T_i) \end{aligned}$$

since  $\rho_{t'}(i) = -\rho_t(i)$  and  $(ts_i)' = t's_i$ . The general case now follows easily by induction on  $\ell(w)$ .  $\square$

By Proposition 1.6(c),  $\mathcal{H}_q(\mathfrak{S}_n)$  is free as an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module. Therefore, the natural induction and restriction are exact functors

$$\text{Ind} : \mathcal{H}_q(\mathfrak{A}_n)\text{-Mod} \rightarrow \mathcal{H}_q(\mathfrak{S}_n)\text{-Mod} \quad \text{and} \quad \text{Res} : \mathcal{H}_q(\mathfrak{S}_n)\text{-Mod} \rightarrow \mathcal{H}_q(\mathfrak{A}_n)\text{-Mod}.$$

between the categories of finitely generated  $\mathcal{H}_q(\mathfrak{S}_n)$ -modules and  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules, respectively. If  $M$  is an  $\mathcal{H}_q(\mathfrak{S}_n)$ -module we abuse notation and write  $M = \text{Res } M$  for the restriction of  $M$  to an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module.

**4.5. Corollary.** *Let  $\lambda$  be a partition. Then  $\tau : S(\lambda) \xrightarrow{\cong} S(\lambda')$  is an isomorphism of  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules.*

*Proof.* By definition  $\tau$  is an isomorphism of  $\mathbb{F}$ -vector spaces, so we only need to show that  $\tau$  commutes with the  $\mathcal{H}_q(\mathfrak{A}_n)$ -action. Suppose that  $a \in \mathcal{H}_q(\mathfrak{A}_n)$  and let  $t \in \text{Std}(\lambda)$ . Then  $\tau(v_{t'} a) = v_{t'} a^\# = v_{t'} a = \tau(v_t) a$  by the Proposition.  $\square$

Let  $\chi^\lambda$  be the character of the  $\mathcal{H}_q(\mathfrak{S}_n)$ -module  $S(\lambda)$  and let  $\chi_{\mathcal{A}}^\lambda$  be the character of the  $\mathcal{H}_q(\mathfrak{A}_n)$ -module  $\text{Res } S(\lambda)$ .

**4.6. Corollary.** *Suppose that  $w \in \mathfrak{S}_n$ . Then  $\chi^\lambda(T_w^\#) = \chi^{\lambda'}(T_w)$ .*

*Proof.* For  $t' \in \text{Std}(\lambda')$  write  $v_{t'} T_w = \sum_{\mathfrak{s}} a_{t'\mathfrak{s}'} v_{\mathfrak{s}'}$  for some scalars  $a_{t'\mathfrak{s}'} = a_{t'\mathfrak{s}'}(w) \in \mathbb{F}$  and where, in the sum,  $\mathfrak{s}' \in \text{Std}(\lambda')$ . Then  $v_t T_w^\# = \sum_{\mathfrak{s}} a_{t'\mathfrak{s}'} v_{\mathfrak{s}}$  by Proposition 4.4. Therefore,  $\chi^\lambda(T_w^\#) = \sum_{t'} a_{t't'} = \chi^{\lambda'}(T_w)$ .  $\square$

Recall from Section 2 that if  $M$  is an  $\mathcal{H}_q(\mathfrak{S}_n)$ -module then  $M^\#$  is the  $\mathcal{H}_q(\mathfrak{S}_n)$  that is equal to  $M$  as a vector space but with the  $\mathcal{H}_q(\mathfrak{S}_n)$ -action twisted by  $\#$ .

**4.7. Corollary.** *Suppose that  $\lambda$  is a partition of  $n$ . Then  $S(\lambda)^\# \cong S(\lambda')$  as  $\mathcal{H}_q(\mathfrak{S}_n)$ -modules.*

Combining Corollary 4.5 and Corollary 4.7,  $S(\lambda) \cong S(\lambda)^\# \cong S(\lambda')$  as  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules. Therefore, a standard application of Clifford theory (Proposition 2.4), implies that  $S(\lambda) \cong S(\lambda')$  is either irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -module if  $\lambda \neq \lambda'$ . It is straightforward to prove this directly by modifying the argument given in Proposition 4.10 below.

The next result shows that if  $\lambda \neq \lambda'$  then the irreducible character  $\chi_{\mathcal{A}}^\lambda$  for  $\mathcal{H}_q(\mathfrak{A}_n)$  is completely determined by the corresponding irreducible character  $\chi^\lambda$  of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Recall from Lemma 1.5 that  $\{A_w \mid w \in \mathfrak{A}_n\}$  is a basis of  $\mathcal{H}_q(\mathfrak{A}_n)$ .

**4.8. Corollary.** *Suppose that  $\lambda$  is a partition of  $n$  and that  $w \in \mathfrak{A}_n$ . Then*

$$\chi_{\mathcal{A}}^\lambda(A_w) = \frac{1}{2}(\chi^\lambda(T_w) + \chi^{\lambda'}(T_w)).$$

It remains to consider the irreducible characters  $\chi_{\mathcal{A}}^{\lambda^\pm}$  of  $\mathcal{H}_q(\mathfrak{A}_n)$ , where  $\lambda = \lambda'$  is a self-conjugate partition of  $n$ . In this case,  $\tau$  is an  $\mathcal{H}_q(\mathfrak{A}_n)$ -module automorphism of  $S(\lambda)$  by Corollary 4.5. A straightforward calculation shows that the eigenspaces of  $\tau$  on  $S(\lambda)$  are  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules. Since  $\tau^2 = 1$ , the only possible eigenvalues for the action of  $\tau$  on  $S(\lambda)$  are  $\pm 1$ .

**4.9. Definition.** *Suppose that  $\lambda = \lambda'$ . Let  $S(\lambda)^+$  and  $S(\lambda)^-$  be the vector subspaces*

$$S(\lambda)^\pm = \text{span}\{v_t \pm v_{t'} \mid t \in \overrightarrow{\text{Std}}(\lambda)\},$$

where  $\overrightarrow{\text{Std}}(\lambda)$  is the set of standard  $\lambda$ -tableau that have 2 in their first row.

**4.10. Proposition.** *Suppose that  $\lambda = \lambda'$ . Then  $S(\lambda)^+$  and  $S(\lambda)^-$  are irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -submodules of  $S(\lambda)$ . Moreover,  $S(\lambda) = S(\lambda)^+ \oplus S(\lambda)^-$  as  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules.*

*Proof.* By inspection,  $S(\lambda)^\pm$  is the  $\pm 1$ -eigenspace for  $\tau$  acting on  $S(\lambda)$ . Therefore,  $S(\lambda)^\pm$  is an  $\mathcal{H}_q(\mathfrak{A}_n)$ -submodule of  $S(\lambda)$  and  $S(\lambda)^+ \cap S(\lambda)^- = 0$ . Now, if  $\mathfrak{t} \in \vec{\text{Std}}(\lambda)$  then either 2 is in the first row of  $\mathfrak{t}$  and  $\mathfrak{t} \in \vec{\text{Std}}(\lambda)$  or 2 is in the first column of  $\mathfrak{t}$  and  $\mathfrak{t}' \in \vec{\text{Std}}(\lambda)$ . Hence,  $S(\lambda) = S(\lambda)^+ \oplus S(\lambda)^-$ . Finally, the two modules  $S(\lambda)^\pm$  are irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules by Clifford theory. Rather than using Clifford theory we can prove this directly using the fact that if  $F_{\mathfrak{t}} \in \mathcal{H}_q(\mathfrak{S}_n)$  is the primitive idempotent attached to the standard  $\lambda$ -tableau  $\mathfrak{t}$ , as in [10, Theorem 3.13], then  $F_{\mathfrak{t}} + F_{\mathfrak{t}^\#} = F_{\mathfrak{t}} + F_{\mathfrak{t}'} \in \mathcal{H}_q(\mathfrak{A}_n)$  since  $F_{\mathfrak{t}^\#} = F_{\mathfrak{t}'}$  by [1, Lemma 2.7]. Consequently, if  $x = \sum_{\mathfrak{s} \in \vec{\text{Std}}(\lambda)(\lambda)} r_{\mathfrak{s}}(v_{\mathfrak{s}} \pm v_{\mathfrak{s}'})$ , for some  $r_{\mathfrak{s}} \in \mathbb{F}$ , is any non-zero element of  $S(\lambda)^\pm$  with  $r_{\mathfrak{t}} \neq 0$  then

$$v_{\mathfrak{t}} \pm v_{\mathfrak{t}'} = \frac{1}{r_{\mathfrak{t}}} x(F_{\mathfrak{t}} + F_{\mathfrak{t}'}) \in x\mathcal{H}_q(\mathfrak{A}_n).$$

Using Proposition 4.2 it follows that  $x\mathcal{H}_q(\mathfrak{A}_n) = S(\lambda)^\pm$ , showing that  $S(\lambda)^\pm$  is an irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -module.  $\square$

Let  $\succ$  be the lexicographic order on  $\mathcal{P}_n$ . The results above show that

$$\{S(\lambda) \mid \lambda \succ \lambda' \text{ for } \lambda \in \mathcal{P}_n\} \cup \{S(\lambda)^\pm \mid \lambda = \lambda' \text{ for } \lambda \in \mathcal{P}_n\}$$

is a complete set of pairwise non-isomorphic absolutely irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules. The classification of the irreducible  $\mathcal{H}_q(\mathfrak{A}_n)$ -modules in the semisimple case is due to Mitsuhashi Proposition 2.4. As we have an explicit realisation of all of the irreducible representations of  $\mathcal{H}_q(\mathfrak{A}_n)$  over  $\mathbb{F}$ , and because these modules cannot split anymore according to Clifford theory, we obtain the following.

**4.11. Corollary.** *The field  $\mathbb{F} = F(\sqrt{-1}, \sqrt{[1]}, \sqrt{[2]}, \dots, \sqrt{[n]})$  is a splitting field for  $\mathcal{H}_{\mathbb{F},q}(\mathfrak{A}_n)$ .*

Let  $\chi_{\mathcal{A}}^{\lambda^\pm}$  be the character of the  $\mathcal{H}_q(\mathfrak{A}_n)$ -module  $S(\lambda)^\pm$ , respectively. We want to compute  $\chi_{\mathcal{A}}^{\lambda^\pm}(a)$ , for  $a \in \mathcal{H}_q(\mathfrak{A}_n)$ . The next result shows that we can reduce the calculation of these characters to what is essentially a computation inside  $\mathcal{H}_q(\mathfrak{S}_n)$ . Before we can state this result we need some more notation.

If  $h \in \mathcal{H}_q(\mathfrak{S}_n)$  then right multiplication by  $h$  gives an endomorphism  $\rho_h$  of  $S(\lambda)$ . Let  $h\tau = \rho_h \circ \tau$  be the endomorphism obtained by composing this map with  $\tau$ . Then  $\chi^\lambda(h\tau)$  is the trace of  $h\tau$  acting on  $S(\lambda)$ .

**4.12. Proposition.** *Suppose that  $\lambda = \lambda'$  and  $a \in \mathcal{H}_q(\mathfrak{A}_n)$ . Then*

$$\chi_{\mathcal{A}}^{\lambda^\pm}(a) = \frac{1}{2}(\chi^\lambda(a) \pm \chi^\lambda(a\tau)).$$

*Proof.* The direct sum decomposition  $S(\lambda) = S(\lambda)^+ \oplus S(\lambda)^-$  implies that  $\chi^\lambda(a) = \chi_{\mathcal{A}}^{\lambda^+}(a) + \chi_{\mathcal{A}}^{\lambda^-}(a)$ , for  $a \in \mathcal{H}_q(\mathfrak{A}_n)$ . Further,  $\{v_{\mathfrak{t}} \pm v_{\mathfrak{t}'} \mid \mathfrak{t} \in \vec{\text{Std}}(\lambda)\}$  is a basis of  $S(\lambda)^\pm$  by Proposition 4.10. Computing traces with respect to this basis shows that  $\chi^\lambda(a\tau) = \chi_{\mathcal{A}}^{\lambda^+}(a) - \chi_{\mathcal{A}}^{\lambda^-}(a)$ . Combining these two formulas,  $\chi_{\mathcal{A}}^{\lambda^\pm}(a) = \frac{1}{2}(\chi^\lambda(a) \pm \chi^\lambda(a\tau))$  as claimed.  $\square$

**4.13. Corollary.** *Suppose that  $\lambda = \lambda'$  and that  $w \in A_n$ . Then*

$$\chi_{\mathcal{A}}^{\lambda^\pm}(A_w) = \frac{1}{2}(\chi^\lambda(T_w) \pm \chi^\lambda(T_w\tau)).$$

*Proof.* By Proposition 4.12,

$$\begin{aligned} \chi_{\mathcal{A}}^{\lambda^\pm}(A_w) &= \frac{1}{4}(\chi^\lambda(T_w + T_w^\#) \pm \chi^\lambda((T_w + T_w^\#)\tau)) \\ &= \frac{1}{2}\chi^\lambda(T_w) \pm \frac{1}{4}(\chi^\lambda(T_w\tau) + \chi^\lambda(T_w^\#\tau)), \end{aligned}$$

by Corollary 4.6 since  $\chi^\lambda(T_w^\#) = \chi^{\lambda'}(T_w) = \chi^\lambda(T_w)$ . Arguing as in Corollary 4.6 shows that  $\chi^\lambda(T_w\tau) = \chi^\lambda(T_w^\#\tau)$ , which completes the proof.  $\square$

As the character values  $\chi^\lambda(T_w)$  are known for  $w \in \mathfrak{S}_n$  (see, for example, [4]), we are reduced to computing  $\chi^\lambda(T_w\tau)$ , for  $w \in A_n$  when  $\lambda = \lambda'$  is a self-conjugate partition. This is the subject of next section.



## 5. THE CHARACTER VALUES FOR SELF-CONJUGATE PARTITIONS

The results of the last section construct the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  and shows that these characters are determined by the irreducible characters of  $\mathcal{H}_q(\mathfrak{S}_n)$  except, possibly, for the characters  $\chi_{\mathcal{A}}^\lambda$  when  $\lambda = \lambda'$ . When  $\lambda = \lambda'$  is a self-conjugate partition [Corollary 4.13](#) reduced the calculation of  $\chi^\lambda(A_w)$  to understanding  $\chi^\lambda(T_w\tau)$ , for  $w \in \mathfrak{A}_n$ . This section computes the character values  $\chi_{\mathcal{H}}^\lambda(T_{w_\kappa}\tau)$  whenever  $\lambda$  is a self-conjugate partition and  $\kappa = (\kappa_1, \dots, \kappa_d)$  is a composition of  $n$ .

The next theorem is one of the main results of this paper. It determines the irreducible characters  $\chi_{\mathcal{A}}^{\lambda\pm}$  of  $\mathcal{H}_q(\mathfrak{A}_n)$  when  $\lambda$  is a self-conjugate partition. Recall that we fixed a field  $\mathbb{F}$  in [Definition 4.1](#) and if  $\lambda = \lambda'$  is self-conjugate then the partition  $h(\lambda) = (h_1, h_2, \dots, h_{d(\lambda)})$  is given in [Definition 3.1](#).

**5.1. Theorem.** *Suppose that  $\lambda = \lambda'$  is a self-conjugate partition of  $n$  and let  $h(\lambda) = (h_1, h_2, \dots, h_{d(\lambda)})$ . Let  $\kappa$  be a composition of  $n$ . Then*

$$\chi^\lambda(T_{w_\kappa}\tau) = \begin{cases} \epsilon_\kappa(-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}, & \text{if } \vec{\kappa} = h(\lambda), \\ 0, & \text{otherwise,} \end{cases}$$

where  $\epsilon_\kappa = (-1)^{\#\{1 \leq y < z \leq d \mid \kappa_y < \kappa_z\}}$ .

In the final section of the paper we show that these character values determine the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  completely. The reader may check that if  $\vec{\kappa} = h(\lambda)$  then  $\ell(w_\kappa) = (n - d(\lambda))/2$ .

Throughout this section we fix a self-conjugate partition  $\lambda = \lambda'$ , and a composition  $\kappa$ , as in [Theorem 5.1](#).

Suppose that  $w \in \mathfrak{S}_n$  and  $\mathfrak{t} \in \text{Std}(\lambda)$ . Define  $\gamma_{\mathfrak{t}}(w) \in \mathbb{F}$  to be the coefficient of  $v_{\mathfrak{t}'}$  in  $v_{\mathfrak{t}}T_w$ , so that

$$v_{\mathfrak{t}}T_w = \gamma_{\mathfrak{t}}(w)v_{\mathfrak{t}'} + \sum_{\substack{\mathfrak{s} \in \text{Std}(\lambda) \\ \mathfrak{s} \neq \mathfrak{t}'}} r_{\mathfrak{s}}v_{\mathfrak{s}},$$

for some  $r_{\mathfrak{s}} \in \mathbb{F}$ . Then  $\chi_{\mathcal{H}}^\lambda(T_w\tau) = \sum_{\mathfrak{t}} \gamma_{\mathfrak{t}}(w)$ , so to determine the character values  $\chi_{\mathcal{H}}^\lambda(T_w\tau)$  it is enough to compute  $\gamma_{\mathfrak{t}}(w)$ , for  $\mathfrak{t} \in \text{Std}(\lambda)$ .

The key definition that we need comes from a paper of Headley [\[8\]](#) who used it to compute the characters of the alternating group  $\mathfrak{A}_n$ . Two integers  $i, j$ , with  $1 \leq i, j \leq n$ , are **diagonally opposite** in  $\mathfrak{t}$  if  $\mathfrak{t}(r, c) = i$  and  $\mathfrak{t}(c, r) = j$ , for some  $(r, c) \in [\lambda]$  with  $r \neq c$ . (Observe that if  $(r, c) \in [\lambda]$  then  $(c, r) \in [\lambda]$  since  $\lambda = \lambda'$ .)

**5.2. Definition** (Headley [\[8, p. 130\]](#)). *Suppose that  $w \in \mathfrak{S}_n$  and that  $\mathfrak{t}$  is a standard  $\lambda$ -tableau, for some partition  $\lambda$ . Then  $\mathfrak{t}$  is **w-transposable** if whenever  $i, j$  are diagonally opposite in  $\mathfrak{t}$  then  $i \in \{j \pm 1\}$  and  $i$  and  $j$  are in the same  $w$ -orbit.*

Let  $\text{Std}(\lambda)_w$  be the set of  $w$ -transposable  $\lambda$ -tableaux.

Let  $\text{Diag}(\mathfrak{t}) = \{\mathfrak{t}(r, r) \mid (r, r) \in [\lambda]\}$  be the set of numbers that lie on the diagonal of the tableau  $\mathfrak{t}$ . If  $\mathfrak{t}$  is  $w$ -transposable and  $i \notin \text{Diag}(\mathfrak{t})$  then  $c_{\mathfrak{t}}(i) = -c_{\mathfrak{t}}(j)$  where  $j \in \{i \pm 1\}$  and  $i$  and  $j$  are diagonally opposite in  $\mathfrak{t}$ . Consequently, the three cases in the next definition are mutually exclusive (and exhaustive).

**5.3. Definition.** *Suppose that  $\lambda$  is a partition of  $n$  and that  $w \in \mathfrak{S}_n$  has a reduced expression of the form  $w = s_{i_1} \dots s_{i_k}$ , where  $1 \leq i_1 < \dots < i_k < n$ . Let  $\mathfrak{t}$  be a  $w$ -transposable standard  $\lambda$ -tableau and, for  $j = 1, \dots, k$ , define*

$$\gamma_{\mathfrak{t}}(i_j) = \begin{cases} \frac{-1}{[\rho_j]}, & \text{if } i_j \in [\mathfrak{t}], \\ \frac{-1}{[\rho_j']}, & \text{if } c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j - 1), \\ \alpha_{\rho_j}, & \text{if } c_{\mathfrak{t}}(i_j) = -c_{\mathfrak{t}}(i_j + 1), \end{cases}$$

where  $\rho_j = \rho_{\mathfrak{t}}(i_j) = c_{\mathfrak{t}}(i_j) - c_{\mathfrak{t}}(i_j + 1)$  and  $\rho_j' = c_{\mathfrak{t}}(i_j - 1) - c_{\mathfrak{t}}(i_j + 1)$ .

Note that if  $w \in \mathfrak{S}_n$  then  $w = s_{i_1} \dots s_{i_k}$ , with  $1 \leq i_1 < \dots < i_k < n$ , if and only if  $w = w_\kappa$  for some composition  $\kappa$  of  $n$ . We have used reduced expressions in [Definition 5.3](#), rather than compositions, only for convenience. See [Example 5.15](#) below for the definition in action.

**5.4. Proposition.** *Suppose that  $w = s_{i_1} \dots s_{i_k}$  is reduced, where  $1 \leq i_1 < \dots < i_k < n$ , and  $\mathfrak{t} \in \text{Std}(\lambda)$ . Then*

$$\gamma_{\mathfrak{t}}(w) = \begin{cases} \gamma_{\mathfrak{t}}(i_1) \dots \gamma_{\mathfrak{t}}(i_k), & \text{if } \mathfrak{t} \text{ is } w\text{-transposable,} \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Observe that  $\gamma_t(w) \neq 0$  (if and) only if there is a sequence  $t_0, t_1, \dots, t_k$  of (not necessarily distinct) standard  $\lambda$ -tableaux such that  $t_0 = t$ ,  $t_k = t'$  and  $v_{t_j}$  appears with non-zero coefficient in  $v_{t_{j-1}}T_{i_j}$ , for  $j = 1, \dots, k$ . By assumption,  $1 \leq i_1 < \dots < i_k < n$  so  $\gamma_t(w) \neq 0$  only if we can change  $t$  into  $t'$  by swapping entries of the form  $i_j$  and  $i_j + 1$ , where  $1 \leq j \leq k$ . Hence,  $\gamma_t(w) \neq 0$  only if  $t$  is  $w$ -transposable, proving the first claim.

Suppose now that  $\gamma_t(w) \neq 0$ . Then the tableaux  $t_0, t_1, \dots, t_k$  above are uniquely determined because  $1 \leq i_1 < \dots < i_k < n$ . Indeed,  $t_j$  is the unique standard  $\lambda$ -tableau that has all of the numbers greater than  $i_j + 1$  in the same positions as they occur in  $t$  and all numbers less than or equal to  $i_j + 1$  in the same positions as in  $t'$ . It follows that if  $v_{t_j}$  appears with coefficient  $\gamma_j$  in  $v_{t_{j-1}}T_{i_j}$  then  $\gamma_t(w) = \gamma_1 \dots \gamma_k$ . To complete the proof it remains to show that  $\gamma_t(i_j) = \gamma_j$  is the coefficient of  $v_{t_j}$  in  $v_{t_{j-1}}T_{i_j}$ . There are three possibilities: either  $i_j$  is on the diagonal of  $t$ , or  $i_j$  is diagonally opposite either  $i_j - 1$  or  $i_j + 1$ .

If  $i_j \in \text{Diag}(t)$  then  $t_j = t_{j-1}$  and, from the definitions,  $\gamma_j = \frac{-1}{[\rho_j]} = \gamma_t(i_j)$ , where  $\rho_j = \rho_{t_j}(i_j)$ .

Next,  $i_j$  is diagonally opposite  $i_j - 1$  if and only if  $c_t(i_j) = -c_t(i_j - 1)$ . In this case we must have  $i_{j-1} = i_j - 1$  and  $t_{j-1} = t_{j-2}s_{i_{j-1}}$  since otherwise we cannot swap  $i_j - 1$  and  $i_j$  in  $t$ . Therefore,  $\gamma_j = \frac{-1}{[\rho'_j]} = \gamma_t(i_j)$ , where  $\rho'_j = c_t(i_j - 1) - c_t(i_j + 1)$ .

Finally,  $i_j$  and  $i_j + 1$  are diagonally opposite in  $t$  if and only if  $c_t(i_j) = -c_t(i_j + 1)$ . In this case we have  $t_j = t_{j-1}s_{i_j}$ , so  $\gamma_j = \alpha_{\rho_j}$ , where  $\rho_j = c_t(i_j) - c_t(i_j + 1)$ . Hence,  $\gamma_j = \gamma_t(i_j)$  as required.  $\square$

**5.5. Corollary.** *Suppose that  $\lambda$  is a partition of  $n$  and that  $w = s_{i_1} \dots s_{i_k}$ , reduced, with  $1 \leq i_1 \leq \dots \leq i_k < n$ . Then*

$$\chi^\lambda(T_w \tau) = \sum_{t \in \text{Std}(\lambda)_w} \gamma_t(i_1) \dots \gamma_t(i_k).$$

*In particular, if there are no  $w$ -transposable  $\lambda$ -tableaux then  $\chi^\lambda(T_w \tau) = 0$ .*

Hence, we need only consider  $w_\kappa$ -transposable tableaux to prove [Theorem 5.1](#).

Recall from [Definition 3.1](#) that if  $\lambda$  is a self-conjugate partition then  $h(\lambda)$  is the partition whose  $i^{\text{th}}$  part is the  $(i, i)$ -hook length of  $\lambda$ . Our first aim is to show that  $\chi^\lambda(T_{w_\kappa} \tau) = 0$  whenever  $\kappa \neq h(\lambda)$ .

**5.6. Corollary** (cf. [8, Lemma 3.3]). *Suppose that  $\kappa$  is a composition of  $n$  such that  $\kappa$  has more than  $d(\lambda)$  parts of odd length. Then  $\chi^\lambda(T_{w_\kappa} \tau) = 0$ .*

*Proof.* Observe that if  $t$  is a  $w$ -transposable tableau then each odd cycle of  $w$  meets the diagonal of  $t$  at least once. Consequently, there are no transposable  $w_\kappa$ -tableaux because  $w_\kappa$  has more than  $d(\lambda)$  odd cycles. Hence,  $\chi^\lambda(T_{w_\kappa} \tau) = 0$  by [Corollary 5.5](#).  $\square$

Before tackling other cycle types we prove a technical Lemma.

**5.7. Lemma.** *Let  $\kappa$  be a composition of  $n$  and suppose that there exists a  $w_\kappa$ -transposable tableau  $t$  and integers  $a$  and  $b$  in  $\text{Diag}(t)$  such  $a < b$  are in the same cycle of  $w_\kappa$  and  $b$  is minimal with this property. Let  $s = ts_{a+1}s_{a+3} \dots s_{b-2}$ . Then  $s$  is  $w_\kappa$ -transposable and  $\gamma_s(w_\kappa) + \gamma_t(w_\kappa) = 0$ .*

*Proof.* Since  $t$  is  $w_\kappa$ -transposable, the numbers  $\{a + 2i + 1, a + 2i + 2\}$ , for  $i = 0, \dots, \frac{b-a-3}{2}$ , are in the same  $w_\kappa$ -orbit and they occupy diagonally opposite positions in  $t$ . Consequently,  $b - a$  is odd and  $s$  is the tableau obtained from  $t$  by swapping these entries. Hence,  $s$  is  $w_\kappa$ -transposable, proving our first claim.

Let  $\gamma_t[a, b] = \prod_{i=a}^{b-1} \gamma_t(i)$  and define  $\gamma_s[a, b]$  similarly. We claim that  $\gamma_t[a, b] = -\gamma_s[a, b]$ . Establishing this claim will prove that  $\gamma_t(w_\kappa) + \gamma_s(w_\kappa) = 0$  because  $\gamma_t(i) = \gamma_s(i)$  if  $i < a$  or if  $i \geq b$ .

As in [Definition 5.3](#), set  $\rho_i = c_t(i) - c_t(i + 1) = c_s(i + 1) - c_s(i)$  and  $\rho'_i = c_t(i - 1) - c_t(i + 1) = c_s(i + 1) - c_s(i - 1)$ , for  $a \leq i < b$ . Since  $c_t(a) = 0 = c_s(a)$ , if  $i = 0, \dots, b - a - 1$  then

$$\gamma_t(a + i) = \begin{cases} \frac{-1}{[-c_t(a+1)]}, & \text{if } i = 0, \\ \frac{-1}{[\rho'_{a+i}]}, & \text{if } i > 0 \text{ is even,} \\ \alpha_{\rho_{a+i}}, & \text{if } i \text{ is odd,} \end{cases}$$

and

$$\gamma_s(a + i) = \begin{cases} \frac{-1}{[-c_s(a+1)]}, & \text{if } i = 0, \\ \frac{-1}{[-\rho'_{a+i}]}, & \text{if } i > 0 \text{ is even,} \\ \alpha_{-\rho_{a+i}}, & \text{if } i \text{ is odd.} \end{cases}$$

Let  $c = \frac{1}{2}(b - a - 1)$  and recall from before [Definition 4.1](#) that  $[-k] = -q^{-2k}[k]$ , for  $k \in \mathbb{Z}$ . Therefore,

$$\begin{aligned}\gamma_t[a, b] &= \frac{(-1)^{c+1}}{[-c_t(a+1)]} \prod_{i=0}^{c-1} \frac{\alpha_{\rho'_{a+2i+1}}}{[\rho'_{a+2i+2}]} = \frac{(-1)^{c+2} q^{2c_t(a+1)}}{[c_t(a+1)]} \prod_{i=0}^{c-1} \frac{q^{-2\rho'_{a+2i+2}} \alpha_{-\rho'_{a+2i+1}}}{[-\rho'_{a+2i+2}]} \\ &= -q^{2(c_t(a+1) - \rho'_{a+2} - \dots - \rho'_{b-1})} \gamma_s[a, b],\end{aligned}$$

since  $c_s(a+1) = -c_t(a+1)$ . The definitions give a collapsing sum,  $c_t(a+1) - \rho'_{a+2} - \dots - \rho'_{b-1} = c_t(b) = 0$ . Hence,  $\gamma_t[a, b] = -\gamma_s[a, b]$ . Therefore, as shown above,  $\gamma_t(w_\kappa) + \gamma_s(w_\kappa) = 0$ , proving the lemma.  $\square$

Recall that  $\ell(\kappa)$  is the number of non-zero parts of  $\kappa$ .

**5.8. Corollary** (cf. [\[8, Lemma 3.4\]](#)). *Let  $\kappa$  be a composition and suppose that  $\ell(\kappa) < d(\lambda)$ . Then  $\chi^\lambda(T_{w_\kappa} \tau) = 0$ .*

*Proof.* Let  $t$  be a  $w_\kappa$ -transposable tableau. As  $w_\kappa$  has  $\ell(\kappa) < d(\lambda)$  cycles, at least one cycle in  $w_\kappa$  meets the diagonal of  $t$  more than once. Therefore, we can find integers  $a < b$  on the diagonal of  $t$ , with  $b$  minimal such that  $a$  and  $b$  are both in the same cycle of  $w_\kappa$ . As in [Lemma 5.7](#) let  $s = ts_{a+1}s_{a+3}\dots s_{b-2}$ . Then  $s$  is the  $w_\kappa$ -transposable obtained by swapping all of the diagonally opposite pairs in  $t$  that occur in the same cycle as  $a$  and  $b$ . Therefore, the map  $t \mapsto s$  gives a pairing of the  $w_\kappa$ -transposable tableaux, so to prove the corollary it is enough to show that  $\gamma_s(w_\kappa) + \gamma_t(w_\kappa) = 0$ . However, this follows from [Lemma 5.7](#).  $\square$

**5.9. Corollary** (cf. [\[8, Lemma 3.5\]](#)). *Suppose that  $\kappa$  is a composition of  $n$  and that  $\kappa$  has at least one even part. Then  $\chi^\lambda(T_{w_\kappa} \tau) = 0$ .*

*Proof.* Let  $t$  be a  $w_\kappa$ -transposable tableau and consider the first even cycle in  $\kappa$ . As  $t$  is transposable, this cycle meets the diagonal of  $t$  in an even number of places. If this cycle meets the diagonal at  $a < b$ , where  $b$  is minimal, then  $\gamma_t(w_\kappa) + \gamma_s(w_\kappa) = 0$  by [Lemma 5.7](#), where  $s = ts_{a+1}\dots s_{b-2}$ . If the first even cycle  $(b-1, b-2, \dots, a+1)$  does not meet the diagonal of  $t$  at all then we again define  $s = ts_{a+1}\dots s_{b-2}$ . By essentially repeating the argument of [Lemma 5.7](#) we find that  $\gamma_t(w_\kappa) + \gamma_s(w_\kappa) = 0$ . Hence, it follows that  $\chi^\lambda(T_{w_\kappa} \tau) = 0$  as required.  $\square$

To prove [Theorem 5.1](#) it remains to consider those elements  $w_\kappa$  that have  $d(\lambda)$  cycles of odd length, and no even cycles. This is by far the most complicated case and it requires some new combinatorial machinery. We begin by reformulating an identity of Greene's [\[5\]](#).

Let  $(X, <)$  be a poset. Then  $X$  is **connected** if its Hasse diagram is connected; otherwise  $X$  is **disconnected**. If  $X$  contains  $m+1$  elements then a **linearisation** of  $X$  is a bijection  $f: X \rightarrow \{0, \dots, m\}$  that respects the ordering in  $X$ ; thus,  $f(x) < f(y)$  whenever  $x < y$  for  $x, y \in X$ . Let  $\mathcal{L}(X)$  be the set of linearisations of  $X$ . If  $f \in \mathcal{L}(X)$  let  $f^{-1}: \{0, \dots, m\} \rightarrow X$  be its inverse. A poset  $(X, <)$  is **semilinear** if, in the Hasse diagram of  $X$ , every vertex has at most two edges. Write  $x \leq y$  if  $x < y$  and  $x$  and  $y$  are adjacent in  $X$  (that is,  $x \leq a \leq y$  only if  $a = x$  or  $a = y$ ).

Let  $X$  be a semilinear poset with  $m+1$  elements. Fix a labelling  $X = \{x_0, x_1, \dots, x_m\}$  of the elements of  $X$  so that  $x_i < x_j$  only if  $j \in \{i \pm 1\}$ . If  $f \in \mathcal{L}(X)$  write  $f^*(i) = j$  if  $f^{-1}(i) = x_j$ , for  $0 \leq i \leq m$ . Define

$$(5.10) \quad \epsilon_X(i) = \begin{cases} 1, & \text{if } x_i \leq x_{i+1}, \\ -1, & \text{if } x_{i+1} \leq x_i, \\ 0, & \text{otherwise,} \end{cases}$$

for  $0 \leq i < m$ , and set  $\epsilon_X = \prod_{i=0}^{m-1} \epsilon_X(i)$ . Then  $X$  is disconnected if and only if  $\epsilon_X = 0$ .

The following result is a mild reformulation of Greene [\[6, Theorem 3.4\]](#).

**5.11. Lemma** (Greene [\[6\]](#)). *Suppose that  $(X, <)$  is a semilinear poset with elements  $\{x_0, \dots, x_m\}$  labelled as above. Suppose that  $\{c_0, \dots, c_m\}$  is a set of pairwise distinct integers. Then*

$$\sum_{f \in \mathcal{L}(X)} q^{2c_{f^*(m)}} \prod_{i=0}^{m-1} \frac{1}{[c_{f^*(i+1)} - c_{f^*(i)}]} = \epsilon_X q^{2c_m} \prod_{i=0}^{m-1} \frac{1}{[c_{i+1} - c_i]}.$$

*Proof.* Let  $\{Q_x \mid x \in X\} = \{Q_0, \dots, Q_m\}$  be a set of indeterminates. Greene [\[6, Theorem 3.4\]](#) has shown that

$$\sum_{f \in \mathcal{L}(X)} \prod_{i=0}^{m-1} \frac{1}{Q_{f^*(i+1)} - Q_{f^*(i)}} = \epsilon_X \prod_{i=0}^{m-1} \frac{1}{Q_{i+1} - Q_i}.$$

To deduce the Lemma from Greene's identity set  $Q_x = q^{2c_x}$ , multiply both sides by  $q^{2(c_0 + \dots + c_m)}(q - q^{-1})^m$  and then simplify each of the factors using the observation that  $\frac{q^{2b}(q - q^{-1})}{q^{2a} - q^{2b}} = \frac{1}{[a-b]}$ , for  $a, b \in \mathbb{Z}$ .  $\square$

5.12. *Remark.* In fact, Greene [6] proves a more general product formula for *planar* posets. When comparing Lemma 5.11 with Greene's theorem note that because  $X$  is semilinear if  $x \neq y \in X$  then the value of Möbius function  $\mu$  at  $(x, y)$  is given by

$$\mu(x, y) = \begin{cases} -1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

The scalar  $\epsilon_X$  does not appear in [6]. We introduce  $\epsilon_X$  as a device for keeping track of the order of the terms on the left-hand side of the identity in Lemma 5.11. These signs are a necessary source of pain below.

We now engineer the circumstances that we need to apply Lemma 5.11.

Until further notice, fix a composition  $\kappa = (\kappa_1, \dots, \kappa_d)$ , such that  $\kappa_1, \dots, \kappa_d$  are all odd and  $d = d(\lambda)$ . Let  $\mathbf{t}$  be a  $w_\kappa$ -transposable tableau. Then  $\mathbf{t}$  determines a sequence of partitions  $\lambda_{\mathbf{t},0} = (0), \lambda_{\mathbf{t},1}, \dots, \lambda_{\mathbf{t},d} = \lambda$  where  $(r, c) \in \llbracket \lambda_{\mathbf{t},z} \rrbracket$  if and only if  $\mathbf{t}(r, c) \leq \kappa_1 + \dots + \kappa_z$ , for  $1 \leq z \leq d$ . That is,  $\lambda_{\mathbf{t},z}/\lambda_{\mathbf{t},z-1}$  is the shape of the skew subtableau of  $\mathbf{t}$  that contains the numbers in cycle  $z$  of  $w_\kappa$ . Each of the partitions  $\lambda_{\mathbf{t},1}, \dots, \lambda_{\mathbf{t},z}$  is symmetric so, in particular,  $(z, z) \in \llbracket \lambda_{\mathbf{t},z} \rrbracket$ , for all  $z$ .

Fix an integer  $z$ , with  $1 \leq z \leq \ell(\kappa) = d$ , and let  $X_{\mathbf{t},z} = \{(r, c) \in \llbracket \lambda_{\mathbf{t},z}/\lambda_{\mathbf{t},z-1} \rrbracket \mid c \geq r\}$ , which we consider as a partially ordered set with  $(r, c) \preceq (r', c')$  if  $r \leq r'$  and  $c \leq c'$ . Set  $k_z = \kappa_1 + \dots + \kappa_{z-1} + 1$  and  $m_z + 1 = |X_{\mathbf{t},z}| = \frac{1}{2}(\kappa_z - 1)$ , so that  $(k_z, k_z + 1, \dots, k_z + 2m_z)$  is the  $z$ th cycle in  $w_\kappa$ . If  $x = (r, c) \in X_{\mathbf{t},z}$  then set  $x' = (c, r)$ . Consequently, if  $x \in X_{\mathbf{t},z}$  then,  $x' \in X_{\mathbf{t},z}$  if and only if  $x = (z, z)$ . Set  $\omega = (z, z)$  and define two closely related *sign sequences*  $\epsilon_{\mathbf{t},z} : X_{\mathbf{t},z} \rightarrow \{\pm 1\}$  and  $\epsilon_{\mathbf{t},z}^\omega : X_{\mathbf{t},z} \rightarrow \{\pm 1\}$  by

$$\epsilon_{\mathbf{t},z}(x) = \begin{cases} 1, & \text{if } \mathbf{t}(x') \geq \mathbf{t}(x), \\ -1, & \text{if } \mathbf{t}(x') < \mathbf{t}(x), \end{cases} \quad \text{and} \quad \epsilon_{\mathbf{t},z}^\omega(x) = \begin{cases} 1, & \text{if } \mathbf{t}(x) \geq \mathbf{t}(\omega), \\ -1, & \text{if } \mathbf{t}(x) < \mathbf{t}(\omega), \end{cases}$$

for  $x \in X_{\mathbf{t},z}$ . By definition,  $\epsilon_{\mathbf{t},z}(\omega) = 1 = \epsilon_{\mathbf{t},z}^\omega(\omega)$  so we could, instead, define the sign sequences to be functions from  $X_{\mathbf{t},z} \setminus \{\omega\} \rightarrow \{\pm 1\}$ . When  $z$  is understood we omit it and simply write  $\epsilon_{\mathbf{t}}$  and  $\epsilon_{\mathbf{t}}^\omega$ . For the tableau  $\mathbf{t}$ , define a linearisation  $f_{\mathbf{t},z} \in \mathcal{L}(X_{\mathbf{t},z})$  of  $X_{\mathbf{t},z}$  by

$$f_{\mathbf{t},z}(x) = \#\{y \in X_{\mathbf{t},z} \mid \mathbf{t}(y) \leq \mathbf{t}(x)\},$$

for  $x \in X_{\mathbf{t},z}$ .

As a prelude to applying Lemma 5.11, the next definition partitions  $\text{Std}(\lambda)_{w_\kappa}$  into more manageable pieces.

5.13. **Definition.** Suppose that  $\kappa$  is a composition of  $n$  and that  $1 \leq z \leq d(\lambda)$ . Let  $\sim_z$  be the equivalence relation on the set of  $w_\kappa$ -transposable  $\lambda$ -tableau determined by  $\mathfrak{s} \sim_z \mathbf{t}$ , for  $\mathfrak{s}, \mathbf{t} \in \text{Std}(\lambda)_{w_\kappa}$ , if:

- a)  $X_{\mathfrak{s},z} = X_{\mathbf{t},z}$  (so that  $\lambda_{\mathfrak{s},z} = \lambda_{\mathbf{t},z}$ ),
- b)  $\mathfrak{s}(x) = \mathbf{t}(x)$ , for all  $x \in \llbracket \lambda \rrbracket / X_{\mathfrak{s},z}$ , and,
- c)  $\epsilon_{\mathfrak{s},z}(x)\epsilon_{\mathfrak{s},z}^\omega(x) = \epsilon_{\mathbf{t},z}(x)\epsilon_{\mathbf{t},z}^\omega(x)$ , for all  $x \in X_{\mathfrak{s},z}$ .

If  $\mathcal{T}$  is a  $\sim_z$ -equivalence class of  $w_\kappa$ -transposable tableaux set  $X_{\mathcal{T}} = X_{\mathbf{t},z}$ ,  $\lambda_{\mathcal{T}} = \lambda_{\mathbf{t},z}$  and

$$\varepsilon_{\mathcal{T}} = \prod_{x \in X_{\mathcal{T}}} \varepsilon_{\mathcal{T}}(x), \quad \text{where} \quad \varepsilon_{\mathcal{T}}(x) = \epsilon_{\mathbf{t},z}(x)\epsilon_{\mathbf{t},z}^\omega(x),$$

and  $\mathbf{t}$  is any element of  $\mathcal{T}$ .

The definition of the  $\sim_z$ -equivalence relation ensures that  $X_{\mathcal{T}}$ ,  $\lambda_{\mathcal{T}}$  and  $\varepsilon_{\mathcal{T}}$  depend only on  $\mathcal{T}$  and not on the choice of  $\mathbf{t} \in \mathcal{T}$ . We warn the reader that, in general, the signs  $\epsilon_{X_{\mathcal{T}}}$  and  $\varepsilon_{\mathcal{T}}$  can be different.

As the next result suggests, the  $\sim_z$ -equivalence classes will allow us to apply Lemma 5.11.

5.14. **Lemma.** Suppose that  $1 \leq z \leq d(\lambda)$  and let  $\mathcal{T}$  be a  $\sim_z$ -equivalence class of  $w_\kappa$ -transposable  $\lambda$ -tableaux. Then the map  $\mathbf{t} \mapsto f_{\mathbf{t},z}$  defines a bijection  $\mathcal{T} \xrightarrow{\sim} \mathcal{L}(X_{\mathcal{T}})$ .

*Proof.* To show that the map  $\mathcal{T} \mapsto \mathcal{L}(X_{\mathcal{T}})$  is a bijection we define an inverse map. First note that if  $\mathfrak{s}, \mathbf{t} \in \mathcal{T}$  then  $\mathfrak{s}$  and  $\mathbf{t}$  agree on  $\lambda \setminus \lambda_{\mathcal{T}}$ . Therefore, if  $f \in \mathcal{L}(X_{\mathcal{T}})$  then to define a tableau  $\mathbf{t}_f \in \mathcal{T}$  we only need to specify its values on  $\lambda/\lambda_{\mathcal{T}}$ . All but one of the numbers  $k_z, k_z + 1, \dots, k_z + 2m_z$  occurs in a diagonally opposite pair, so the requirement that  $\varepsilon_{\mathcal{T}}(x) = \epsilon_{\mathbf{t}_f}(x)\epsilon_{\mathbf{t}_f}^\omega(x)$  is constant on  $\mathcal{T}$  implies that there is a unique tableau  $\mathbf{t}_f \in \mathcal{T}$  such that  $\mathbf{t}_f(\omega) = k_z + 2f(\omega)$  and if  $x \in X_{\mathcal{T}}$  and  $x \neq \omega = (z, z)$  then

$$\{\mathbf{t}_f(x), \mathbf{t}_f(x')\} = \begin{cases} \{k_z + 2f(x), k_z + 2f(x) + 1\}, & \text{if } f(x) < f(\omega), \\ \{k_z + 2f(x) - 1, k_z + 2f(x)\}, & \text{if } f(x) > f(\omega). \end{cases}$$

The maps  $\mathbf{t} \mapsto f_{\mathbf{t}}$  and  $f \mapsto \mathbf{t}_f$  are mutually inverse, so  $\mathcal{T} \xrightarrow{\sim} \mathcal{L}(X_{\mathcal{T}})$  as claimed.  $\square$

The following example should help the reader absorb all of the new definitions.

**5.15. Example.** Suppose that  $\lambda = (6, 3, 2, 1^3)$ , so that  $h(\lambda) = (11, 3)$  and take  $\kappa = (7^2)$ . Then  $w_\kappa = s_1 \dots s_6 s_8 \dots s_{13}$  and the reader may check that there are 384  $w_\kappa$ -transposable  $\lambda$ -tableaux. We take  $z = 2$ , so that  $k_z = 8$  and  $m_z = 3$ , and consider the  $\sim_z$ -equivalence class  $\mathcal{T}$  of  $w_\kappa$ -transposable tableaux with  $\lambda_{t,z} = \lambda / (4, 1^3)$  and with sign sequence  $\varepsilon_{\mathcal{T}} = - - +$ . That is,  $\epsilon_{t,z}(x_1)\epsilon_{t,z}^\omega(x_2) = -1$ ,  $\epsilon_{t,z}(x_2)\epsilon_{t,z}^\omega(x_3) = -1$  and  $\epsilon_{t,z}(x_3)\epsilon_{t,z}^\omega(d) = 1$ , where  $x_0 = (2, 2) = \omega$ ,  $x_1 = (2, 3)$ ,  $x_2 = (1, 5)$  and  $x_3 = (1, 6)$ . We use this shorthand for all sign sequences in the table below.

t	$\epsilon_{t,2}$	$\epsilon_{t,2}^\omega$	$f_{t,2}^*$	$\gamma_t(8)$	$\gamma_t(9)$	$\gamma_t(10)$	$\gamma_t(11)$	$\gamma_t(12)$	$\gamma_t(13)$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 12 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 8 10</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">14</div> </div>	---	+++	0123	$\frac{-1}{[0+1]}$	$\alpha_2$	$\frac{-1}{[-1+4]}$	$\alpha_8$	$\frac{-1}{[-4-5]}$	$-\alpha_{10}$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 10 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 8 12</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">14</div> </div>	---	+++	0213	$\frac{-1}{[0+4]}$	$\alpha_8$	$\frac{-1}{[-4+1]}$	$\alpha_2$	$\frac{-1}{[-1-5]}$	$-\alpha_{10}$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 10 11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 8 14</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">12</div> </div>	---	+++	0312	$\frac{-1}{[0+4]}$	$\alpha_8$	$\frac{-1}{[-4-5]}$	$-\alpha_{10}$	$\frac{-1}{[5+1]}$	$\alpha_2$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 8 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 10 12</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">14</div> </div>	-++	+-+	1203	$-\alpha_8$	$\frac{-1}{[4-0]}$	$\frac{-1}{[0+1]}$	$\alpha_2$	$\frac{-1}{[-1-5]}$	$-\alpha_{10}$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 8 11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 10 14</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">12</div> </div>	-++	+-+	1302	$-\alpha_8$	$\frac{-1}{[4-0]}$	$\frac{-1}{[0-5]}$	$-\alpha_{10}$	$\frac{-1}{[5+1]}$	$\alpha_2$
<div style="display: flex; flex-direction: column; align-items: center;"> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· · · · 8 11</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 12 14</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">· 13</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">·</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">9</div> <div style="border: 1px solid black; padding: 2px; margin-bottom: 2px;">10</div> </div>	-+-	+--	2301	$-\alpha_8$	$\frac{-1}{[4+5]}$	$\alpha_{10}$	$\frac{-1}{[-5-0]}$	$\frac{-1}{[0+1]}$	$\alpha_2$

In the tableaux above we have only indicated the positions of the numbers 8, ..., 14 because these are the only entries that will matter in the arguments below (because the tableaux in a  $\sim_z$ -equivalence class are constant on the nodes outside of  $X_{\mathcal{T}}$ ).  $\diamond$

Now fix a  $\sim_z$ -equivalence class of tableaux  $\mathcal{T}$  with poset  $X_{\mathcal{T}}$ , where  $1 \leq z \leq d(\lambda)$ . Define

$$(5.16) \quad \alpha(X_{\mathcal{T}}) = \prod_{x \in X_{\mathcal{T}} \setminus \{\omega\}} \alpha_{2c(x)}.$$

For a  $w_\kappa$ -transposable tableau  $t$  set

$$(5.17) \quad \gamma_{t,z}(w_\kappa) = \prod_{j=0}^{2m_z-1} \gamma_t(k_z + j).$$

Then,  $\gamma_t(w_\kappa) = \gamma_{t,1}(w_\kappa) \dots \gamma_{t,d}(w_\kappa)$ . The reader may find it helpful to consult [Example 5.15](#) above during the proofs of the next few results.

**5.18. Lemma.** Suppose that  $\kappa = (\kappa_1, \dots, \kappa_d)$  is a composition of  $n$  such that  $d = d(\lambda)$  and  $\kappa_1, \dots, \kappa_d$  are all odd. Fix  $1 \leq z \leq d$  and let  $\mathcal{T}$  be a  $\sim_z$ -equivalence class of  $w_\kappa$ -transposable tableaux and  $\mathbf{t} \in \mathcal{T}$ . Then

$$\gamma_{\mathbf{t},z}(w_\kappa) = \frac{\alpha(X_{\mathcal{T}}) \prod_{i=0}^{m_z} \epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})}{\prod_{i=0}^{m_z-1} [c_{\mathbf{t}}(i) - c_{\mathbf{t}}(i+1)]},$$

where  $x_i^{\mathbf{t}} = f_{\mathbf{t},z}^{-1}(i)$  and  $c_{\mathbf{t}}(i) = \epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}})$ , for  $i = 0, \dots, m_z$ .

*Proof.* By definition,  $\gamma_{\mathbf{t},z}(w_\kappa) = \prod_{j=0}^{2m_z-1} \gamma_{\mathbf{t}}(k_z + j)$  and  $\alpha(X_{\mathcal{T}}) = \prod_{x \in X_{\mathcal{T}}} \alpha_{2c(x)}$ . So, we need to show that

$$\prod_{j=0}^{2m_z-1} \gamma_{\mathbf{t}}(k_z + j) = \prod_{x \in X_{\mathcal{T}}} \epsilon_{\mathbf{t}}(x) \alpha_{2c(x)} \prod_{i=0}^{m_z-1} \frac{1}{[\epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}}) - \epsilon_{\mathbf{t}}(x_{i+1}^{\mathbf{t}})c(x_{i+1}^{\mathbf{t}})]}.$$

There are three mutually exclusive cases to consider, corresponding to the different cases in [Definition 5.3](#).

Fix  $j$ , with  $1 \leq j \leq 2m_z$ , and let  $x$  and  $y$  be the unique nodes in  $X_{\mathbf{t}}$  such that  $k_z + j \in \{\mathbf{t}(x), \mathbf{t}(x')\}$  and  $k_z + j + 1 \in \{\mathbf{t}(y), \mathbf{t}(y')\}$ , where we allow and, in fact, need to include the possibility that  $\{x, x'\} = \{y, y'\}$ .

*Case 1.*  $k_z + j \in \text{Diag}(\mathbf{t})$ . Therefore,  $x = (z, z) = \omega$  and  $c_{\mathbf{t}}(k_z + j) = 0$ . Write  $x = x_i^{\mathbf{t}}$ , so that  $y = x_{i+1}^{\mathbf{t}}$ . Now,  $\mathbf{t}(y) \geq \mathbf{t}(x) + 1$  if and only if  $\epsilon_{\mathbf{t}}(y) = 1$ , in which case  $c_{\mathbf{t}}(k_z + j + 1) = c(x_{i+1}^{\mathbf{t}})$ . Similarly,  $\mathbf{t}(y') \geq \mathbf{t}(x) + 1$  if and only if  $\epsilon_{\mathbf{t}}(y) = -1$  in which case  $c_{\mathbf{t}}(k_z + j + 1) = -c(x_{i+1}^{\mathbf{t}})$ . Hence, in both cases,  $c_{\mathbf{t}}(k_z + j + 1) = \epsilon_{\mathbf{t}}(x_{i+1}^{\mathbf{t}})c(x_{i+1}^{\mathbf{t}})$ . Therefore,

$$\gamma_{\mathbf{t}}(k_z + j) = \frac{-1}{[c_{\mathbf{t}}(k_z + j) - c_{\mathbf{t}}(k_z + j + 1)]} = \frac{-1}{[\epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}}) - \epsilon_{\mathbf{t}}(x_{i+1}^{\mathbf{t}})c(x_{i+1}^{\mathbf{t}})]}.$$

*Case 2.*  $c_{\mathbf{t}}(k_z + j) = -c_{\mathbf{t}}(k_z + j - 1)$ . In this case  $k_z + j$  and  $k_z + j - 1$  are diagonally opposite. Hence, if  $x = x_{i+1}^{\mathbf{t}}$  then  $y = x_i^{\mathbf{t}}$ . Further,  $\mathbf{t}(x) = k_z + j - 1$  if and only if  $\epsilon_{\mathbf{t}}(x) = 1$ , and  $\mathbf{t}(y) = k_z + j - 1$  if and only if  $\epsilon_{\mathbf{t}}(y) = 1$ . Hence, as in Case 1,

$$\gamma_{\mathbf{t}}(k_z + j) = \frac{-1}{[c_{\mathbf{t}}(k_z + j - 1) - c_{\mathbf{t}}(k_z + j + 1)]} = \frac{-1}{[\epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}}) - \epsilon_{\mathbf{t}}(x_{i+1}^{\mathbf{t}})c(x_{i+1}^{\mathbf{t}})]}.$$

*Case 3.*  $c_{\mathbf{t}}(k_z + j) = -c_{\mathbf{t}}(k_z + j + 1)$ . In this case  $k_z + j$  and  $k_z + j + 1$  are diagonally opposite so  $x = y$  and, in particular,  $x \neq (z, z)$ . Write  $x = x_i^{\mathbf{t}}$ . Then

$$\gamma_{\mathbf{t}}(k_z + j) = \alpha_{c_{\mathbf{t}}(k_z + j + 1) - c_{\mathbf{t}}(k_z + j)} = \alpha_{-2\epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}})} = -\epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})\alpha_{2c(x_i^{\mathbf{t}})},$$

where the last equality uses the fact that  $\alpha_{-c} = -\alpha_c$ , for  $c \in \mathbb{Z}$ .

By construction,  $0 \leq i < m_z$  in Cases 1–3 with  $\omega \in \{x_i^{\mathbf{t}}, x_{i+1}^{\mathbf{t}}\}$  only in Case 1. Cases 1 and 2 contribute the same quantity, as a function of  $i$ , to  $\gamma_{\mathbf{t},z}(w_\kappa)$ . There are  $2m_z$  minus signs in Cases 1–3, so the lemma now follows by combining these three cases since  $\prod_{x \neq \omega} \epsilon_{\mathbf{t}}(x) = \prod_{x \in X_{\mathcal{T}}} \epsilon_{\mathbf{t}}(x)$ .  $\square$

As before [Lemma 5.11](#), fix a labelling  $X_{\mathcal{T}} = \{x_0, x_1, \dots, x_{m_z}\}$  that is compatible with the partial order on  $X_{\mathcal{T}}$  in the sense that  $x_i < x_j$  only if  $j \in \{i \pm 1\}$ . For  $0 \leq i \leq m_z$  define

$$c_{\mathcal{T}}(i) = \varepsilon_{\mathcal{T}}(x_i)c(x_i) \in \mathbb{Z}.$$

We are now ready to compute  $\sum_{\mathbf{t} \in \mathcal{T}} \gamma_{\mathbf{t},z}(w_\kappa)$ .

**5.19. Proposition.** Suppose that  $\kappa = (\kappa_1, \dots, \kappa_d)$  is a composition of  $n$  that has  $d = d(\lambda)$  odd parts. Fix  $z$  with  $1 \leq z \leq d(\lambda)$  and let  $\mathcal{T}$  be a  $\sim_z$ -equivalence class of  $w_\kappa$ -transposable  $\lambda$ -tableaux. Then

$$\sum_{\mathbf{t} \in \mathcal{T}} \gamma_{\mathbf{t},z}(w_\kappa) = \varepsilon_{\mathcal{T}} \prod_{x \in X_{\mathcal{T}}} (-1)^{m_z} q^{2c_{\mathcal{T}}(m_z)} \alpha(X_{\mathcal{T}}) \prod_{i=0}^{m_z-1} \frac{1}{[c_{\mathcal{T}}(i+1) - c_{\mathcal{T}}(i)]}.$$

*Proof.* As in [Lemma 5.18](#), for  $\mathbf{t} \in \mathcal{T}$  let  $c_{\mathbf{t}}(i) = \epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}})$ , where  $x_i^{\mathbf{t}} = f_{\mathbf{t}}^{-1}(i)$  and  $0 \leq i \leq m_z$ . Then

$$\sum_{\mathbf{t} \in \mathcal{T}} \gamma_{\mathbf{t}}(w_\kappa) = \alpha(X_{\mathcal{T}}) \sum_{\mathbf{t} \in \mathcal{T}} \frac{\prod_{i=0}^{m_z} \epsilon_{\mathbf{t}}(x_i)}{\prod_{i=0}^{m_z-1} [c_{\mathbf{t}}(i) - c_{\mathbf{t}}(i+1)]}.$$

by [Lemma 5.18](#). Now, if  $\mathbf{t} \in \mathcal{T}$  then  $x_i^{\mathbf{t}} = x_j$ , where  $j = f_{\mathbf{t}}^*(i)$ . So, using [Definition 5.13\(c\)](#),

$$c_{\mathbf{t}}(i) = \epsilon_{\mathbf{t}}(x_i^{\mathbf{t}})c(x_i^{\mathbf{t}}) = \epsilon_{\mathbf{t}}^\omega(x_j)\epsilon_{\mathbf{t}}^\omega(x_j)\epsilon_{\mathbf{t}}(x_j)c(x_j) = \epsilon_{\mathbf{t}}^\omega(x_j)c_j^{\mathcal{T}} = \epsilon_{\mathbf{t}}^\omega(x_i^{\mathbf{t}})c_{\mathcal{T}}(f_{\mathbf{t}}^*(i)).$$



By definition,  $\epsilon_t^\omega(x_i^\dagger) = -1$  if and only if  $0 \leq i < f_t(\omega)$ . Moreover,  $\epsilon_t(x) = \varepsilon_T(x)\epsilon_t^\omega(x)$ , for  $x \in X_T$ . Therefore, using the fact that  $c(\omega) = 0$  for the second and fourth equalities,

$$\begin{aligned}
\sum_{t \in T} \gamma_t(w_\kappa) &= \alpha(X_T) \sum_{t \in T} \frac{\prod_{i=0}^{m_z} \varepsilon_T(x_i) \epsilon_t^\omega(x_i)}{\prod_{i=0}^{m_z-1} [\epsilon_t^\omega(x_i) c_T(f_t^*(i)) - \epsilon_t^\omega(x_{i+1}^\dagger) c_T(f_t^*(i+1))]} \\
&= \varepsilon_T \alpha(X_T) \sum_{t \in T} \prod_{i=0}^{f_t(\omega)-1} \frac{-1}{[c_T(f_t^*(i+1)) - c_T(f_t^*(i))]} \prod_{i=f_t(\omega)}^{m_z-1} \frac{1}{[c_T(f_t^*(i)) - c_T(f_t^*(i+1))]} \\
&= \varepsilon_T \alpha(X_T) \sum_{t \in T} \prod_{i=0}^{f_t(\omega)-1} \frac{-1}{[c_T(f_t^*(i+1)) - c_T(f_t^*(i))]} \prod_{i=f_t(\omega)}^{m_z-1} \frac{-q^{2(c_T(f_t^*(i+1)) - c_T(f_t^*(i)))}}{[c_T(f_t^*(i+1)) - c_T(f_t^*(i))]} \\
&= \varepsilon_T (-1)^{m_z} \alpha(X_T) \sum_{t \in T} q^{2c_T(f_t^*(m_z))} \prod_{i=0}^{m_z-1} \frac{1}{[c_T(f_t^*(i+1)) - c_T(f_t^*(i))]} \\
&= \varepsilon_T (-1)^{m_z} \alpha(X_T) \sum_{f \in \mathcal{L}(X_T)} q^{2c_T(f^*(m_z))} \prod_{i=0}^{m_z-1} \frac{1}{[c_T(f^*(i+1)) - c_T(f^*(i))]},
\end{aligned}$$

where the last equation invokes the bijection  $T \xrightarrow{\sim} \mathcal{L}(X_T)$  of Lemma 5.14. Applying Lemma 5.11 now completes the proof.  $\square$

If the poset  $X_T$  is disconnected then  $\epsilon_{X_T} = 0$  so that  $\sum_{t \in T} \gamma_{t,z}(w_\kappa) = 0$  by Proposition 5.19. We therefore need to determine when  $X_T$  is connected.

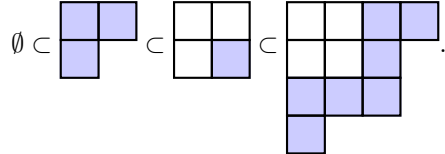
Let  $\lambda$  and  $\mu$  be partitions with  $\mu \subset \lambda$ . Then  $\lambda/\mu$  is a **strip** if the multiset  $\{c(x) \mid x \in \llbracket \lambda/\mu \rrbracket\}$  contains distinct consecutive integers.

**5.20. Definition.** Suppose that  $\kappa = (\kappa_1, \dots, \kappa_d)$  is a composition of  $n$ . A **symmetric covering** of  $\lambda$  of **type**  $\kappa$  is a sequence of self-conjugate partitions

$$\lambda^{(0)} = (0) \subset \lambda^{(1)} \subset \dots \subset \lambda^{(d)} = \lambda$$

such that  $\lambda^{(z)}/\lambda^{(z-1)}$  is a strip and  $|\lambda^{(z)}| = \kappa_1 + \dots + \kappa_z$ , for  $1 \leq z \leq d$ . The composition  $\kappa$  **symmetrically covers**  $\lambda$  if a symmetric covering of  $\lambda$  of type  $\kappa$  exists.

For example, if  $\lambda = (4, 3, 3, 1)$  and  $\kappa = (3, 1, 7)$  then the following diagrams show that  $\kappa$  symmetrically covers  $\lambda$ :



Notice that  $h(\lambda) = (7, 3, 1)$  so that  $\vec{\kappa} = h(\lambda)$ .

**5.21. Lemma.** Suppose that  $\kappa$  is a composition of  $n$  such that  $\ell(\kappa) = d(\lambda)$ . Then  $\kappa$  symmetrically covers  $\lambda$  if and only if  $\vec{\kappa} = h(\lambda)$ . Moreover, if  $\vec{\kappa} = h(\lambda)$  then there is a unique symmetric covering of  $\lambda$  of type  $\kappa$ .

*Proof.* Write  $\kappa = (\kappa_1, \dots, \kappa_d)$ , where  $d = d(\lambda) = \ell(\kappa)$ . We argue by induction on  $d$ . If  $d = 1$  then  $\kappa$  symmetrically covers  $\lambda$  if and only if  $n$  is odd,  $\kappa = (n) = h(\lambda)$  and  $\lambda = (\frac{n+1}{2}, 1^{\frac{n-1}{2}})$  is a hook partition. Now suppose that  $d > 1$  and write  $h(\lambda) = (h_1, \dots, h_d)$ .

Suppose that  $\kappa$  symmetrically covers  $\lambda$ . Then  $\nu = (\kappa_1, \dots, \kappa_{d-1})$  symmetrically covers  $\lambda^{(d-1)}$  so that  $\vec{\nu} = h(\lambda^{(d-1)})$  by induction. As  $\lambda = \lambda^{(d)}$  and  $\lambda^{(d-1)}$  are both symmetric and  $\lambda^{(d)}/\lambda^{(d-1)}$  is connected, it follows that  $\kappa_d$  is odd and that  $\kappa_d = h_i$  where  $i$  is minimal such that  $\lambda_i \neq \lambda_i^{(d-1)}$ . Hence,  $\vec{\kappa} = h(\lambda)$ .

Conversely, suppose that  $d > 1$  and that  $\vec{\kappa} = h(\lambda)$  and let  $\nu = (\kappa_1, \dots, \kappa_{d-1})$ . Then  $\kappa_d = h_i$  for some  $i$ , so that  $\vec{\nu} = h(\mu) = (h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_d)$ , where the diagram of  $\mu$  is obtained by deleting the  $(i, i)$ -rim hook from the diagram of  $\lambda$  (the  $(i, i)$ -rim hook is the set of nodes along the rim of  $\lambda$  that connect the nodes  $(i, \lambda_i)$  and  $(\lambda_i, i)$ ).

Finally, the uniqueness of the covering is immediate from the construction in the last paragraph.  $\square$

Combining Lemma 5.21 with Proposition 5.19 we can prove that the characters  $\chi^\lambda(T_{w_\kappa} \tau)$  vanish when  $\kappa$  has  $d(\lambda)$  odd parts and  $\vec{\kappa} \neq h(\lambda)$ .

**5.22. Corollary.** *Let  $\kappa = (\kappa_1, \dots, \kappa_d)$  be a composition of  $n$  that has  $d = d(\lambda)$  non-zero parts, all of which are odd, and suppose that  $\vec{\kappa} \neq h(\lambda)$ . Then  $\chi_{\mathcal{H}}^\lambda(T_{w_\kappa} \tau) = 0$ .*

*Proof.* By Corollary 5.5 it is enough to show that  $\sum_{\mathbf{t}} \gamma_{\mathbf{t}}(w_\kappa) = 0$ , where  $\mathbf{t}$  runs over the set of  $w_\kappa$ -transposable  $\lambda$ -tableaux. Let  $\mathbf{t}$  be a  $w_\kappa$ -transposable  $\lambda$ -tableau. By assumption  $\vec{\kappa} \neq h(\lambda)$  so, by Lemma 5.21, we can find an integer  $z$  such that  $1 \leq z \leq d(\lambda)$  and  $\lambda_{\mathbf{t},z}/\lambda_{\mathbf{t},z-1}$  is not connected. Let  $z$  be the smallest such integer and let  $\mathcal{T}$  be the  $\sim_z$ -equivalence class of  $w_\kappa$ -transposable  $\lambda$ -tableaux that contains  $\mathbf{t}$ . In this way we attach a pair  $(z, \mathcal{T})$  to each  $w_\kappa$ -transposable  $\lambda$ -tableau (different  $z$ 's can appear for different tableaux). The equivalence classes that we obtain in this way partition the set of  $w_\kappa$ -transposable  $\lambda$ -tableaux because the choices of  $\sim_z$ -equivalence classes are determined by the cycles of  $w_\kappa$  and the minimality of  $z$ . Hence, it is enough to show that  $\sum_{\mathbf{t} \in \mathcal{T}} \gamma_{\mathbf{t}}(w_\kappa) = 0$  whenever  $\mathcal{T}$  is one of the chosen  $\sim_z$ -equivalence classes of  $w_\kappa$ -transposable  $\lambda$ -tableau. The poset  $X_{\mathcal{T}}$  is disconnected by the choice of  $z$ . Therefore,  $\epsilon_{X_{\mathcal{T}}} = 0$  and so  $\sum_{\mathbf{t} \in \mathcal{T}} \gamma_{\mathbf{t}}(w_\kappa) = 0$  by Proposition 5.19.  $\square$

We have now computed  $\chi^\lambda(T_{w_\kappa} \tau)$  in all cases except for when  $\vec{\kappa} = h(\lambda)$ . Fix a composition  $\kappa$  such that  $\vec{\kappa} = h(\lambda)$  together with the symmetric covering  $\lambda^{(0)} \subset \dots \subset \lambda^{(d)}$  of  $\lambda$  of type  $\kappa$  given by Lemma 5.21.

**5.23. Corollary.** *Suppose that  $\kappa$  is a composition of  $n$  such that  $\vec{\kappa} = h(\lambda)$  and that  $\mathbf{t} \in \text{Std}(\lambda)_{w_\kappa}$  is a  $w_\kappa$ -transposable tableau. Then  $\lambda_{\mathbf{t},z} = \lambda^{(z)}$ , for  $1 \leq z \leq d$ . In particular,  $X_{\mathbf{t},z}$  is connected.*

*Proof.* By Lemma 5.21 there is a unique sequence of self-conjugate partitions  $\lambda^{(0)}, \dots, \lambda^{(d)}$  such that  $|\lambda^{(z)}| = \kappa_1 + \dots + \kappa_z$  and  $\lambda^{(z)}/\lambda^{(z-1)}$  is connected, for  $1 \leq z \leq d$ . On the other hand, if  $\mathbf{t}$  is a  $w_\kappa$ -transposable  $\lambda$ -tableau then  $\lambda_{\mathbf{t},1}, \dots, \lambda_{\mathbf{t},d}$  is a symmetric covering of  $\lambda$ . Therefore,  $\lambda_{\mathbf{t},z} = \lambda^{(z)}$ , for  $1 \leq z \leq d$ .  $\square$

By Corollary 5.23, if  $\mathbf{t}$  is a  $w_\kappa$ -transposable tableau then the partitions  $\lambda_{\mathbf{t},z}$  depends only on  $\lambda$ ,  $\kappa$  and  $z$  and not on  $\mathbf{t}$ . Let  $\mathbf{t}_{\kappa,z}$  be the restriction of  $\mathbf{t}$  to  $\lambda^{(z)}/\lambda^{(z-1)}$ . Then the entries of  $\mathbf{t}_{\kappa,z}$  are precisely the integers  $\{k_z + i \mid 0 \leq i \leq 2m_z\}$ . Let

$$\text{Std}(\lambda)_{\kappa,z} = \{\mathbf{t}_{\kappa,z} \mid \mathbf{t} \in \text{Std}(\lambda)_{w_\kappa}\}.$$

Then  $\text{Std}(\lambda)_{\kappa,z}$  is the set of skew tableau of shape  $\lambda^{(z)}/\lambda^{(z-1)}$  that are  $w_\kappa$ -transposable in the obvious sense. The sets  $\text{Std}(\lambda)_{\kappa,z}$  are compatible with the different  $\sim_y$ -equivalence classes in the sense that if  $\mathbf{s} \sim_z \mathbf{t}$  then  $\mathbf{s}_{\kappa,y} = \mathbf{t}_{\kappa,y}$  for  $y \neq z$ . Hence, by Corollary 5.23,

$$(5.24) \quad \text{Std}(\lambda)_{w_\kappa} \xrightarrow{\sim} \prod_{z=1}^{d(\lambda)} \text{Std}(\lambda)_{\kappa,z},$$

where the bijection is given by  $\mathbf{t} \mapsto (\mathbf{t}_{\kappa,1}, \dots, \mathbf{t}_{\kappa,d})$ . Abusing notation slightly, we restrict the equivalence relation  $\sim_z$  to  $\text{Std}(\lambda)_{\kappa,z}$  and let  $\text{Std}[\lambda]_{\kappa,z}$  be the set of  $\sim_z$ -equivalence classes of skew tableaux in  $\text{Std}(\lambda)_{\kappa,z}$ .

Recall that we have fixed a labelling  $\{x_0, \dots, x_{m_z}\}$  of  $X_{\mathcal{T}}$  such that if  $x_i < x_j$  then  $j \in \{i \pm 1\}$ . As  $X_{\mathcal{T}}$  is a connected strip, there are exactly two such labellings (unless  $m_z = 0$ , of course). We fix the labelling of  $X_{\mathcal{T}}$  so that  $x_i$  is the unique node in  $X_{\mathcal{T}}$  such that  $c(x_i) = i$ , for  $0 \leq i \leq m_z$ . In particular,  $x_0 = \omega = (z, z)$ .

The next identity, which is quite cute, is the last piece of the puzzle that we need to compute  $\chi^\lambda(T_{w_\kappa} \tau)$ .

**5.25. Lemma.** *Suppose that  $\vec{\kappa} = h(\lambda)$  and  $1 \leq z \leq d(\lambda)$ . Then*

$$\sum_{\mathcal{T} \in \text{Std}[\lambda]_{\kappa,z}} \epsilon_{\mathcal{T}} q^{2c_{\mathcal{T}}(m_z)} \prod_{i=0}^{m_z-1} \frac{1}{[c_{\mathcal{T}}(i+1) - c_{\mathcal{T}}(i)]} = q^{-m_z} \prod_{i=1}^{m_z} \frac{[2i]}{[2i-1]}.$$

*Proof.* For clarity we write  $m = m_z$  throughout the proof. We want to argue by induction on  $m$  but, before we can do this, we first need a more explicit description of  $\text{Std}[\lambda]_{\kappa,z}$ . Recall the sign sequence  $\epsilon_{\mathcal{T}}$  from Definition 5.13 and set  $\underline{\epsilon}_{\mathcal{T}} = (\epsilon_{\mathcal{T}}(x_0), \dots, \epsilon_{\mathcal{T}}(x_m))$ . We claim that the map  $\mathcal{T} \mapsto \underline{\epsilon}_{\mathcal{T}}$  defines a bijection

$$\text{Std}[\lambda]_{\kappa,z} \xrightarrow{\sim} \mathcal{E}_m = \{(1, \epsilon_1, \dots, \epsilon_m) \mid \epsilon_i = \pm 1\}.$$

(Consequently,  $\#\text{Std}[\lambda]_{\kappa,z} = 2^m$ .) To see this first note that if  $\mathcal{T}$  is a  $\sim$ -equivalence class then  $\epsilon_{\mathcal{T}}(x_0) = 1$ . Therefore, by Definition 5.13(c) and Corollary 5.23 there are at most  $2^m$  different  $\sim$ -equivalence classes of tableaux in  $\text{Std}[\lambda]_{\kappa,z}$ . On the other hand, if  $\mathbf{t}$  is any  $w_\kappa$ -transposable tableau then we obtain  $2^m$  different  $w_\kappa$ -transposable tableaux, each of which is in a different  $\sim$ -equivalence class, by swapping all pairs  $\{\mathbf{t}(x), \mathbf{t}(x')\}$  of diagonally opposite entries in  $\mathbf{t}$  in all possible ways, for  $x \in X_{\mathcal{T}} \setminus \{\omega\}$ . This establishes the claim.

Identifying  $\text{Std}[\lambda]_{\kappa,z}$  and  $\mathcal{E}_m$  via the bijection above, we now argue by induction on  $m$ . If  $m = 0$  then both sides of the identity that we want to prove are equal to 1, so there is nothing to prove (by convention empty products are 1). We now assume that  $m > 1$  and, by induction, that the lemma holds for  $\mathcal{E}_m$ . Let  $\mathcal{T}$  and  $\mathcal{T}'$  be

equivalence classes in  $\mathcal{E}_{m+1}$  with  $\varepsilon_{\mathcal{T}'}(x_r) = \varepsilon_{\mathcal{T}}(x_r)$ , for  $0 \leq r \leq m$ , and  $\varepsilon_{\mathcal{T}}(x_{m+1}) = 1 = -\varepsilon_{\mathcal{T}'}(x_{m+1})$ . The contribution that  $\mathcal{T}$  and  $\mathcal{T}'$  make to the sum over  $\mathcal{E}_{m+1}$  is

$$\varepsilon_{\mathcal{T}} q^{2c_{\mathcal{T}}(m)} \left( \frac{q^{2(c_{\mathcal{T}}(m+1)-c_{\mathcal{T}}(m))}}{[c_{\mathcal{T}}(m+1)-c_{\mathcal{T}}(m)]} - \frac{q^{2(-c_{\mathcal{T}}(m+1)-c_{\mathcal{T}}(m))}}{[-c_{\mathcal{T}}(m+1)-c_{\mathcal{T}}(m)]} \right) \prod_{i=0}^{m-1} \frac{1}{[c_{\mathcal{T}}(i+1)-c_{\mathcal{T}}(i)]}.$$

By [Corollary 5.23](#), the posets  $X_{\mathcal{T}}$  and  $X_{\mathcal{T}'}$  are both connected so  $c_{\mathcal{T}}(m+1) = m+1 = -c_{\mathcal{T}'}(m+1)$  and  $c_{\mathcal{T}}(m) = c_{\mathcal{T}'}(m) = \varepsilon_{\mathcal{T}}(x_m)m = \pm m$ . If  $\varepsilon_{\mathcal{T}}(x_m) = 1$  then  $c_{\mathcal{T}}(m) = m$  and

$$\frac{q^{2(m+1-c_{\mathcal{T}}(m))}}{[m+1-c_{\mathcal{T}}(m)]} - \frac{q^{-2(1+m+c_{\mathcal{T}}(m))}}{[-1-m-c_{\mathcal{T}}(m)]} = \frac{q^2}{[1]} - \frac{q^{-2(1+2m)}}{[-1-2m]} = q^{-1} \frac{[2m+2]}{[2m+1]}.$$

Similarly if  $\varepsilon_{\mathcal{T}}(x_m) = -1$  then essentially the same calculation shows that the difference of these two terms is again  $q^{-1}[2m+2]/[2m+1]$ . All of the sequences in  $\mathcal{E}_{m+1}$  occur in pairs of the form  $(\underline{\varepsilon}_{\mathcal{T}}, \underline{\varepsilon}_{\mathcal{T}'})$ , so

$$\begin{aligned} \sum_{\underline{\varepsilon}_{\mathcal{T}} \in \mathcal{E}_{m+1}} \varepsilon_{\mathcal{T}} q^{c_{\mathcal{T}}(m+1)} \prod_{i=0}^m \frac{1}{[c_{\mathcal{T}}(i+1)-c_{\mathcal{T}}(i)]} &= q^{-1} \frac{[2m+2]}{[2m+1]} \sum_{s \in \mathcal{E}_{m-1}} \varepsilon_s q^{c_s(m)} \prod_{i=0}^m \frac{1}{[c_s(i+1)-c_s(i)]} \\ &= q^{-(m+1)} \prod_{i=1}^{m+1} \frac{[2i]}{[2i-1]} \end{aligned}$$

where the last equality follows by induction. This completes the proof of the inductive step and the lemma.  $\square$

We can now prove [Theorem 5.1](#), which is arguably the most important result in this paper.

*Proof of Theorem 5.1.* Recall that  $h(\lambda) = (h_1, \dots, h_{d(\lambda)})$ . We need to show that

$$\chi^{\lambda}(T_{w_{\kappa}}\tau) = \begin{cases} \epsilon_{\kappa}(-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}, & \text{if } \vec{\kappa} = h(\lambda), \\ 0, & \text{otherwise,} \end{cases}$$

The results in this section show that  $\chi^{\lambda}(T_{w_{\kappa}}\tau) = 0$  if  $\kappa$  is a composition of  $n$  such that  $\vec{\kappa} \neq h(\lambda)$ ; see [Corollary 5.6](#), [Corollary 5.8](#), [Corollary 5.9](#) and [Corollary 5.22](#). It remains to compute  $\chi^{\lambda}(T_{w_{\kappa}}\tau)$  when  $\vec{\kappa} = \lambda$ . Using [Corollary 5.5](#), [\(5.24\)](#) and [Proposition 5.19](#),

$$\begin{aligned} \chi_{\mathcal{H}}^{\lambda}(T_{w_{\kappa}}\tau) &= \sum_{t \in \text{Std}(\lambda)_{\kappa, z}} \prod_{z=1}^{d(\lambda)} \gamma_{t, z}(w_{\kappa}) = \prod_{z=1}^{d(\lambda)} \sum_{\mathcal{T} \in \text{Std}[\lambda]_{\kappa, z}} \sum_{t \in \mathcal{T}} \gamma_{t, z}(w_{\kappa}) \\ &= \prod_{z=1}^{d(\lambda)} \sum_{\mathcal{T} \in \text{Std}[\lambda]_{\kappa, z}} (-1)^{m_z} q^{2c_{\mathcal{T}}(m_z)} \varepsilon_{\mathcal{T}} \alpha(X_{\mathcal{T}}) \prod_{i=0}^{m_z-1} \frac{1}{[c_{\mathcal{T}}(i+1)-c_{\mathcal{T}}(i)]}. \end{aligned}$$

Fix  $z$  with  $1 \leq z \leq d(\lambda)$ . By [Corollary 5.23](#), if  $\mathcal{T} \in \text{Std}[\lambda]_{\kappa, z}$  then  $X_{\mathcal{T}} = X_z$  depends only on  $\kappa$  and  $z$  (and  $\lambda$ ). Similarly, the sign  $\varepsilon_{X_z} = \varepsilon_{X_{\mathcal{T}}}$  depends only on  $\kappa$  and  $z$ . Expanding the definition of  $\alpha(X_{\mathcal{T}})$ , from [\(5.16\)](#),

$$\alpha(X_{\mathcal{T}}) = \prod_{x \in X_{\mathcal{T}} \setminus \{\omega\}} \alpha_{2c(x)} = \prod_{j=0}^{m_z} \alpha_{2j} = (\sqrt{-1})^{m_z} \frac{\sqrt{[2m_z+1]}}{\sqrt{[1]}} \prod_{i=1}^{m_z} \frac{[2i-1]}{[2i]}.$$

Consequently,  $\alpha(X_z) = \alpha(X_{\mathcal{T}})$  also depends only on  $\kappa_z$ . Set  $\epsilon'_{\kappa} = \prod_{z=1}^{d(\lambda)} \epsilon_{X_z}$ . Then the equation above becomes

$$\begin{aligned} \chi_{\mathcal{H}}^{\lambda}(T_{w_{\kappa}}\tau) &= \epsilon'_{\kappa} \prod_{z=1}^{d(\lambda)} (-1)^{m_z} \alpha(X_{\mathcal{T}}) \sum_{\mathcal{T} \in \text{Std}[\lambda]_{\kappa, z}} \varepsilon_{\mathcal{T}} q^{2c_{\mathcal{T}}(m_z)} \prod_{i=0}^{m_z-1} \frac{1}{[c_{\mathcal{T}}(i+1)-c_{\mathcal{T}}(i)]} \\ &= \epsilon'_{\kappa} \prod_{z=1}^{d(\lambda)} (-1)^{m_z} \alpha(X_{\mathcal{T}}) q^{-m_z} \prod_{i=1}^{m_z} \frac{[2i]}{[2i-1]}, & \text{by Lemma 5.25,} \\ &= \epsilon'_{\kappa} (-q^{-1}\sqrt{-1})^{m_1+\dots+m_z} \prod_{z=1}^{d(\lambda)} \frac{\sqrt{[2m_z+1]}}{\sqrt{[1]}}, \end{aligned}$$

where the last equality uses the formula for  $\alpha(X_z)$  given above. Now,  $\sqrt{[1]} = \sqrt{q}$ ,  $\sum_{z=1}^{d(\lambda)} m_z = \frac{1}{2}(n - d(\lambda))$  and  $\{h_1, \dots, h_{d(\lambda)}\} = \{2m_z + 1 \mid 1 \leq z \leq d(\lambda)\}$ . Therefore, we have shown that

$$\chi_{\mathcal{H}}^{\lambda}(T_{w_{\kappa}}\tau) = \epsilon'_{\kappa}(-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}.$$

To complete the proof it remains to show that  $\epsilon'_{\kappa} = \epsilon_{\kappa}$ , where  $\epsilon_{\kappa}$  is the sign defined in [Theorem 5.1](#). By [\(5.10\)](#),  $\epsilon_{X_z} = (-1)^{\rho(z)}$  where  $\rho(z) = \#\{0 \leq i < m_z \mid x_{i+1} > x_i\}$ . Therefore,  $\rho(z) + 1$  is equal to the number of different rows in  $[\lambda] \cap X_z$ . More precisely, if  $x_{m_z} = (r, c)$  then  $\rho(z) = z - r$ . Armed with this observation it follows that if  $\kappa_y < \kappa_{y+1}$  then  $\epsilon'_{\kappa} = -\epsilon'_{\pi}$  where  $\pi = (\kappa_1, \dots, \kappa_{y+1}, \kappa_y, \dots, \kappa_d)$  is obtained from  $\kappa$  by swapping  $\kappa_y$  and  $\kappa_{y+1}$ . It follows that  $\epsilon'_{\kappa} = (-1)^{\#\{1 \leq y < z \leq d(\lambda) \mid \kappa_y < \kappa_z\}} = \epsilon_{\kappa}$  as required. The theorem is proved.  $\square$

## 6. COMPUTING ALL CHARACTER VALUES OF $\mathcal{H}_q(\mathfrak{A}_n)$

Combining [Corollary 4.8](#), [Corollary 4.13](#) and [Theorem 5.1](#) we can compute the values of all of the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  on the elements  $A_{w_{\kappa}}$ , whenever  $\kappa$  is a composition of  $n$ . Now  $\{w_{\kappa} \mid \kappa \in \mathcal{P}_n\}$  is a set of minimal length conjugacy class representatives for  $\mathfrak{S}_n$ , however, it is not a complete set of minimal length conjugacy class representatives for  $\mathfrak{A}_n$  because it does not contain the elements  $w_{\kappa}^{-}$  whenever  $\kappa$  is a partition of  $n$  with distinct odd parts. Consequently, if  $\lambda = \lambda'$  then we do not yet know the value of the characters  $\chi^{\lambda \pm}(A_{w_{\kappa}^{-}})$  when  $\kappa$  is a composition of  $n$  with distinct odd parts. More importantly, if  $\chi$  is a character of  $\mathcal{H}_q(\mathfrak{A}_n)$  then we do not know how to compute  $\chi(A_w)$  if  $w \neq w_{\kappa}^{\pm}$  for some composition  $\kappa$ . This section shows that the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  are determined by characters of  $\mathcal{H}_q(\mathfrak{S}_n)$  and [Theorem 5.1](#).

The conjugacy classes of  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$  have been described in [Section 3](#). It is well-known that the characters of  $\mathcal{H}_q(\mathfrak{S}_n)$  are not class functions in the sense that if  $\chi$  is a character then, in general,  $\chi(T_v)$  and  $\chi(T_w)$  are not necessarily equal if  $v$  and  $w$  are conjugate in  $\mathfrak{S}_n$ . For example, this already happens for the character of the trivial character because  $1_{\mathcal{H}}(T_w) = q^{\ell(w)}$ , for  $w \in \mathfrak{S}_n$ . Nonetheless, as we now recall, the characters of  $\mathcal{H}_q(\mathfrak{S}_n)$  are determined by their values on a set of minimal length conjugacy class representatives.

For each conjugacy class  $C \in \mathcal{C}(\mathfrak{S}_n)$  let  $C_{\min} = \{x \in C \mid \ell(x) \leq \ell(y) \text{ for all } y \in C\}$  be the set of minimal length conjugacy class representatives. Fix an element  $w_C \in C_{\min}$  for each conjugacy class. For example, if  $C = C_{\kappa}$  is the  $\mathfrak{S}_n$ -conjugacy class of permutations of cycle type  $\kappa \in \mathcal{P}_n$  then we could set  $w_C = w_{\kappa}$ .

**6.1. Theorem** (Geck-Pfeiffer [[4](#), §8.2]). *Suppose that  $\chi$  is a character of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Then there exist Laurent polynomials  $\{f_{C,w}(q) \in \mathbb{Z}[q, q^{-1}] \mid w \in \mathfrak{S}_n \text{ and } C \in \mathcal{C}(\mathfrak{S}_n)\}$ , which do not depend on  $\chi$ , such that*

$$\chi(T_w) = \sum_{C \in \mathcal{C}(\mathfrak{S}_n)} f_{C,w}(q) \chi(T_{w_C})$$

for any character  $\chi$  of  $\mathcal{H}_q(\mathfrak{S}_n)$ . Moreover, if  $C \in \mathcal{C}(\mathfrak{S}_n)$  then  $f_{C,w}(q) = 0$  if  $\ell(w) < \ell(w_C)$  and  $\chi(T_x) = \chi(T_{w_C})$  for all  $x \in C_{\min}$ .

The polynomials  $\{f_{C,w}(q)\}$  are the **class polynomials** of  $\mathcal{H}_q(\mathfrak{S}_n)$ .

Recall the basis  $\{B_w \mid w \in \mathfrak{A}_n\}$  of  $\mathcal{H}_q(\mathfrak{A}_n)$  from [Proposition 1.7](#) together with the involutions  $\bar{\phantom{x}}$ , on  $\mathcal{Z}$  and  $\epsilon$ , on  $\mathcal{H}_{\mathbb{Z},q}(\mathfrak{S}_n)$ , from [Corollary 1.10](#). Extend these involutions to  $\mathbb{F}$  and  $\mathcal{H}_q(\mathfrak{S}_n) = \mathcal{H}_{\mathbb{F},q}(\mathfrak{S}_n)$  by setting

$$\overline{\sqrt{-1}} = \sqrt{-1} \quad \text{and} \quad \overline{\sqrt{[k]}} = \sqrt{-1} \sqrt{[-k]}, \quad \text{for } k > 0.$$

The argument of [Corollary 1.10](#) applies without change over  $\mathbb{F}$ .

**6.2. Lemma.** *Suppose that  $\lambda \in \mathcal{P}_n$  and  $w \in \mathfrak{S}_n$ . Then  $\chi^{\lambda}(B_w) = \varepsilon_w \overline{\chi^{\lambda}(B_w)}$ .*

*Proof.* Write  $v_s B_w = \sum_{\mathfrak{t}} b_{\mathfrak{st}}(w) v_s$ , for some  $b_{\mathfrak{st}}(w) \in \mathcal{Z}$ . Now  $\epsilon(B_w) = \varepsilon_w B_w$ , by [Corollary 1.10](#), so we must have  $b_{\mathfrak{st}}(w) = \varepsilon_w b_{\mathfrak{st}}(w)$ . Taking traces proves the lemma.  $\square$

**6.3. Lemma.** *Suppose that  $\lambda \in \mathcal{P}_n$  and  $w \in \mathfrak{S}_n \setminus \mathfrak{A}_n$ . Then  $(\chi^{\lambda} + \chi^{\lambda'})(B_w) = 0$ .*

*Proof.* By [Corollary 4.6](#),  $\chi^{\lambda'}(B_w) = \chi^{\lambda}(B_w^{\#})$  and  $B_w^{\#} = -B_w$  since  $w \notin \mathfrak{A}_n$ . Therefore,

$$(\chi^{\lambda} + \chi^{\lambda'})(B_w) = \chi^{\lambda}(B_w) + \chi^{\lambda'}(B_w) = \chi^{\lambda}(B_w) + \chi^{\lambda}(B_w^{\#}) = \chi^{\lambda}(B_w) - \chi^{\lambda}(B_w) = 0.$$

$\square$

The analogues of the last two lemmas are false for the  $A$ -basis of  $\mathcal{H}_q(\mathfrak{S}_n)$ .

We can now prove an analogue of the Geck-Pfeiffer theorem for the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  that are indexed by partitions that are not self-conjugate.

**6.4. Proposition.** Suppose that  $\lambda \in \mathcal{P}_n$  and  $w \in \mathfrak{A}_n$ . Then there exist polynomials  $g_{C,w}(q) \in \mathcal{Z}$ , which are independent of  $\lambda$ , such that

$$\chi_{\mathcal{A}}^\lambda(A_w) = \sum_{C \in \mathcal{C}(\mathfrak{A}_n)} g_{C,w}(q) \chi_{\mathcal{A}}^\lambda(A_{w_C}).$$

Moreover,  $\chi_{\mathcal{A}}^\lambda(A_x) = \chi_{\mathcal{A}}^\lambda(A_{w_C})$  whenever  $x \in C_{\min}$ .

*Proof.* We argue by induction on  $\ell(w)$ . If  $\ell(w) = 1$  then  $w = 1$  and  $\chi_{\mathcal{A}}^\lambda(A_w) = \dim S(\lambda)$  and there is nothing to prove. Suppose then that  $\ell(w) > 0$  and that  $w \in D \in \mathcal{C}(\mathfrak{S}_n)$ . Recall from [Corollary 4.8](#) that

$$\chi_{\mathcal{A}}^\lambda(A_w) = \frac{1}{2}(\chi^\lambda(T_w) + \chi^{\lambda'}(T_w)).$$

Therefore, if  $w \in D_{\min}$  then  $\chi_{\mathcal{A}}^\lambda(A_w) = \chi_{\mathcal{A}}^\lambda(A_{w_D})$  by [Theorem 6.1](#). In particular,  $\chi_{\mathcal{A}}^\lambda(A_w) = \chi_{\mathcal{A}}^\lambda(A_{w_D})$  whenever  $w \in D_{\min}$  (and, in fact,  $\chi_{\mathcal{A}}^\lambda(w)$  depends only on the  $\mathfrak{S}_n$  conjugacy class of  $w$  and not the  $\mathfrak{A}_n$ -conjugacy class). Hence, the proposition follows when  $w \in D_{\min}$  since  $D$  is a disjoint union of  $\mathfrak{A}_n$  conjugacy classes.

Now suppose that  $w$  is not of minimal length in its conjugacy class. Applying [Theorem 6.1](#),

$$\chi_{\mathcal{A}}^\lambda(A_w) = \sum_{C \in \mathcal{C}(\mathfrak{S}_n)} \frac{1}{2} f_{C,w}(\chi^\lambda(T_{w_C}) + \chi^{\lambda'}(T_{w_C})).$$

For  $y \in \mathfrak{S}_n$  we can write  $T_y = \sum_{y \leq v} s_{yv} B_y$ , for some  $s_{yv} \in \mathcal{Z}$ . Therefore,

$$\begin{aligned} \chi_{\mathcal{A}}^\lambda(A_w) &= \sum_{y \in \mathfrak{S}_n} \left( \sum_{\substack{C \in \mathcal{C}(\mathfrak{S}_n) \\ y \leq w_C}} \frac{1}{2} s_{yw_C} f_{C,w} \right) (\chi^\lambda + \chi^{\lambda'})(B_y) \\ &= \sum_{y \in \mathfrak{A}_n} \left( \sum_{\substack{C \in \mathcal{C}(\mathfrak{S}_n) \\ y \leq w_C}} s_{yw_C} f_{C,w} \right) \chi_{\mathcal{A}}^\lambda(B_y) \end{aligned}$$

where the last equality follows from [Lemma 6.3](#) and [Corollary 4.6](#). Note that in the sum  $y \in \mathfrak{A}_n$ . By [Proposition 1.7](#), if  $y \in \mathfrak{A}_n$  then  $B_y = \sum_{x \leq y} r_{xy} A_x$ , for some  $r_{xy} \in \mathcal{Z}$  such that  $r_{xy} \neq 0$  only if  $x \in \mathfrak{A}_n$ . Hence, the sum above becomes

$$\chi_{\mathcal{A}}^\lambda(A_w) = \sum_{x \in \mathfrak{A}_n} \left( \sum_{\substack{y \in \mathfrak{A}_n \\ x \leq y}} \sum_{\substack{C \in \mathcal{C}(\mathfrak{S}_n) \\ y \leq w_C}} r_{xy} s_{yw_C} f_{C,w} \right) \chi_{\mathcal{A}}^\lambda(A_x).$$

Since  $w$  is not of minimal length in its conjugacy class  $f_{C,w} \neq 0$  only if  $\ell(w_C) < \ell(w)$ . Consequently,  $\chi_{\mathcal{A}}^\lambda(A_x)$  contributes to  $\chi_{\mathcal{A}}^\lambda(A_w)$  only if  $\ell(x) \leq \ell(y) \leq \ell(w_C) < \ell(w)$  in the last displayed equation. Therefore, by induction on length,  $\chi_{\mathcal{A}}^\lambda(A_x)$  can be written in the required form. It follows that  $\chi_{\mathcal{A}}^\lambda(A_w)$  can be written in the required form, so the proof is complete.  $\square$

[Proposition 6.4](#) implies that if  $\chi$  is a character of  $\mathcal{H}_q(\mathfrak{S}_n)$  then the restriction of  $\chi$  to  $\mathcal{H}_q(\mathfrak{A}_n)$  is determined by the character values  $\{\chi(A_{w_C}) \mid C \in \mathcal{C}(\mathfrak{A}_n)\}$  on the minimal length conjugacy classes of  $\mathfrak{A}_n$ . In particular, if  $\lambda \neq \lambda'$  then  $\chi_{\mathcal{A}}^\lambda$  is an irreducible character of  $\mathcal{H}_q(\mathfrak{A}_n)$  and  $\chi_{\mathcal{A}}^\lambda(A_w)$  is determined by [Proposition 6.4](#), for  $w \in \mathfrak{A}_n$ .

We now consider the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  that are not the restriction of a character of  $\mathcal{H}_q(\mathfrak{S}_n)$ . For this it is enough to consider the irreducible characters  $\chi_{\mathcal{A}}^{\lambda^\pm}$  of  $\mathcal{H}_q(\mathfrak{A}_n)$ , where  $\lambda$  is a self-conjugate partition of  $n$ . To compute  $\chi_{\mathcal{A}}^{\lambda^\pm}(A_w)$ , for  $w \in \mathfrak{A}_n$ , we need to delve deeper into the proof of [Theorem 6.1](#).

Following [\[3\]](#), let  $\xrightarrow{s}$  be the transitive closure of the relation  $\xrightarrow{s}$  on  $\mathfrak{S}_n$ , for  $s \in S$  where if  $x, y \in \mathfrak{S}_n$  then  $x \xrightarrow{s} y$  if  $y = sxs$  and  $\ell(y) \leq \ell(x)$ . Secondly, let  $\approx$  be the equivalence relation on  $\mathfrak{S}_n$  generated by the relations  $\xrightarrow{u}$ , for  $u \in \mathfrak{S}_n$ , where  $x \xrightarrow{u} y$  if  $\ell(x) = \ell(y)$  and either  $ux = yu$  and  $\ell(ux) = \ell(u) + \ell(x)$ , or  $xu = yu$  and  $\ell(xu) = \ell(x) + \ell(u)$ .

We state the next important theorem only in the special case of the symmetric groups but, using case-by-case arguments, Geck and Pfeiffer [\[3\]](#) proved this result for any finite Coxeter group. He and Nie [\[7\]](#) have generalised this result to the extended affine Weyl groups using an elegant case-free proof.

**6.5. Theorem** (Geck and Pfeiffer [\[3\]](#)). Let  $C$  be a conjugacy class of  $\mathfrak{S}_n$ .

- a) If  $x \in C$  then there exists an element  $y \in C_{\min}$  such that  $x \xrightarrow{s} y$ .
- b) If  $x, y \in C_{\min}$  then  $x \approx y$ .

We will use [Theorem 5.1](#) and [Theorem 6.5](#) to determine the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$ . Given [Theorem 6.5](#), it is straightforward to prove [Theorem 6.1](#). In contrast, it will take quite a bit of work to compute the character values  $\chi^\lambda(T_w\tau)$ , for  $w \in \mathfrak{S}_n$ . The complication is that we are not able to make use of [Theorem 6.5\(b\)](#) because when we try to apply it to compute  $\chi(T_w\tau)$  then terms  $\chi(T_v\tau)$  with  $\ell(v) > \ell(w)$  can appear and this breaks any argument that uses induction on the length of  $w$ . Fortunately, [Theorem 5.1](#) allows  $\kappa$  to be a composition, rather than just a partition, so we can replace [Theorem 6.5\(b\)](#) with the next result.

**6.6. Lemma.** *Let  $w \in C_{\min}$ , where  $C$  is an  $\mathfrak{S}_n$ -conjugacy class. Then  $w \rightarrow w_\kappa$ , for some composition  $\kappa$  of  $n$ .*

*Proof.* As remarked above, the elements of  $C_{\min}$  are Coxeter elements in some standard parabolic subgroup. When  $x$  is written as a product of disjoint cycles the different cycles commute, so it is enough to show if  $x$  is a Coxeter element in  $\mathfrak{S}_{n+1}$  then  $x \rightarrow w_{(n+1)} = s_1 \dots s_n$ . As  $x$  is a Coxeter element in  $\mathfrak{S}_{n+1}$  we can write  $x = s_{i_1} \dots s_{i_n}$ , where  $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ . For  $r = 0, 1, \dots, n$  we claim that  $x \rightarrow s_1 \dots s_{r-1} s_{j_r} \dots s_{j_n}$  for some  $j_r, \dots, j_n$  such that  $\{j_r, \dots, j_n\} = \{r, r+1, \dots, n\}$ . To prove this we argue by induction on  $r$ . If  $r = 0$  then there is nothing to prove because we can take  $j_t = i_t$ , for  $1 \leq t \leq n$ . Hence, by induction, we may assume that  $x \rightarrow s_1 \dots s_{r-1} s_{j_r} \dots s_{j_n}$  for suitable  $j_r, \dots, j_n$ . By assumption,  $j_k = r$  for some  $k \geq r$ . If  $k = r$  the inductive assumption automatically holds. If  $k > r$  then, since  $s_{j_r}, \dots, s_{j_{k-1}}$  commute with  $s_1, \dots, s_{r-1}$ ,

$$\begin{aligned} x &\xrightarrow{s_{j_r}} s_{j_r} x s_{j_r} = s_1 \dots s_{r-1} s_{j_{r+1}} \dots s_r s_{j_{k+1}} \dots s_{j_n} s_{j_r} \\ &\xrightarrow{s_{j_{r+1}}} s_{j_{r+1}} s_{j_r} x s_{j_r} s_{j_{r+1}} \xrightarrow{s_{j_{r+2}}} \dots \\ &\xrightarrow{s_{j_{k-1}}} s_{j_{k-1}} \dots s_{j_r} x s_{j_r} \dots s_{j_{k-1}} = s_1 \dots s_r s_{j_{k+1}} \dots s_{j_n} s_{j_r} \dots s_{j_{k-1}}, \end{aligned}$$

proving the inductive step. This proves the claim and hence the lemma.  $\square$

We need some preparatory lemmas before we can apply these results to compute  $\chi^\lambda(A_w\tau)$ , for an arbitrary permutation  $w \in \mathfrak{A}_n$ . Recall that  $\tau$  is the unique  $\mathbb{F}$ -linear endomorphism of  $S(\lambda)$  such that  $v_t\tau = v_{t'}$ , for  $t \in \text{Std}(\lambda)$ .

**6.7. Lemma.** *Suppose that  $\lambda$  is a self-conjugate partition and  $w \in \mathfrak{S}_n$ . Then*

$$v\tau T_w = vT_w^\# \tau \quad \text{and} \quad vT_w\tau = v\tau T_w^\#,$$

for all  $v \in S(\lambda)$ .

*Proof.* By linearity, it is enough to consider the case when  $v = v_t$ , for  $t \in \text{Std}(\lambda)$ . Then  $v_t T_w^\# = v_t \tau T_w \tau$ , by [Proposition 4.4](#), implying the result.  $\square$

Recall that  $S = \{s_1, \dots, s_{n-1}\}$  is the set of Coxeter generators of  $\mathfrak{S}_n$ . If  $s \in S$  and  $w \in \mathfrak{S}_n$  then  $\ell(sws) - \ell(w) \in \{0, \pm 2\}$ . We need the following well-known result that is proved, for example, in [\[14, Lemma 1.9\]](#).

**6.8. Lemma.** *Suppose that  $s \in S$  and  $w \in \mathfrak{S}_n$  such that  $\ell(sws) = \ell(w)$  and  $\ell(sw) = \ell(ws)$ . Then  $w = sws$ .*

Hence, if  $sws \neq w$  and  $\ell(w) = \ell(sws)$  then either  $\ell(ws) > \ell(w) = \ell(sws) > \ell(sw)$  or  $\ell(sw) > \ell(w) = \ell(sws) > \ell(ws)$ . The next lemma relates the character values  $\chi^\lambda(T_x\tau)$  and  $\chi^\lambda(T_y\tau)$  whenever  $x \rightarrow y$ . It would be better if we could prove an analogue of this result for the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$ , unfortunately, it is not clear how to do this.

**6.9. Lemma.** *Suppose that  $\lambda$  is a partition of  $n$  and suppose that  $w \in \mathfrak{S}_n$  and  $s \in S$ .*

a) *Suppose that  $w \neq sws$  and  $\ell(w) = \ell(sws)$ . Then*

$$\chi^\lambda(T_w\tau) = -\chi^\lambda(T_{sws}\tau) + (q - q^{-1})\chi^\lambda(T_v\tau),$$

where  $v$  is uniquely determined by the requirements that  $\ell(v) < \ell(w)$  and  $v \in \{sw, ws\}$ .

b) *If  $\ell(w) > \ell(sws)$  then  $\chi^\lambda(T_w\tau) = -\chi^\lambda(T_{sws}\tau)$ .*

*Proof.* First consider (a). Now  $v = ws$  if and only if  $\ell(w) > \ell(ws)$ , in which case [Lemma 6.7](#) implies that

$$\begin{aligned} \chi^\lambda(T_w\tau) &= \chi^\lambda(T_{ws}T_s\tau) = \chi^\lambda(T_{ws}\tau T_s^\#) = \chi^\lambda(T_{ws}\tau(-T_s + q - q^{-1})) \\ &= -\chi^\lambda(T_s T_{ws}\tau) + (q - q^{-1})\chi^\lambda(T_{ws}\tau), \end{aligned}$$

giving the result since  $T_s T_{ws} = T_{sws}$ . The argument when  $v = sw$  is similar.

For part (b), by [Lemma 6.7](#),  $\chi^\lambda(T_w\tau) = \chi^\lambda(T_s T_{sws} T_s\tau) = \chi^\lambda(T_{sws}\tau T_s^\# T_s) = -\chi^\lambda(T_{sws}\tau)$ , where the last equality follows because  $T_s^\# = -T_s^{-1}$ .  $\square$



Recall from [Section 3](#) that the  $\mathfrak{S}_n$ -conjugacy class  $C_\kappa$  splits into two  $\mathfrak{A}_n$ -conjugacy classes if and only if the parts of  $\kappa$  are distinct and all odd. In this case  $w_\kappa^+$  and  $w_\kappa^- = s_r w_\kappa^+ s_r$ , for some  $r$ , are minimal length  $\mathfrak{A}_n$ -conjugacy class representatives.

**6.10. Corollary.** *Suppose that  $\lambda = \lambda'$  is a partition of  $n > 1$  and that  $\kappa$  is composition of  $n$  with  $\vec{\kappa} = h(\lambda)$ . Then*

$$\chi^\lambda(T_{w_\kappa^-}) = -\chi^\lambda(T_{w_\kappa^+}).$$

*Proof.* By [Lemma 6.9\(a\)](#),  $\chi^\lambda(T_{w_\kappa^-}\tau) = -\chi^\lambda(T_{w_\kappa}\tau) + (q - q^{-1})\chi^\lambda(T_v\tau)$  where  $v$  is the unique permutation in  $\{s_r w_\kappa^-, w_\kappa^- s_r\}$  with  $\ell(v) < \ell(w_\kappa^-) = \ell(w_\kappa)$ . By [Theorem 6.5\(a\)](#), if  $C$  is the  $\mathfrak{S}_n$ -conjugacy class containing  $v$  then  $v \rightarrow w_C$  for some  $w_C \in C_{\min}$ . The  $\rightarrow$  relation is generated by the two situations considered in [Lemma 6.9](#), so it follows that  $\chi^\lambda(T_v\tau)$  can be written as a  $\mathbb{Z}$ -linear combination of character values  $\chi^\lambda(T_{w_D}\tau)$ , where the sum is over  $D \in \mathcal{C}(\mathfrak{S}_n)$  with  $w_D \in D_{\min}$  and  $\ell(w_D) \leq \ell(v)$ . By [Lemma 6.6](#), and possibly further applications of [Lemma 6.9](#), we can assume that  $w_D = w_\sigma$ , for some composition  $\sigma$ . If  $\sigma$  is a composition appearing in this way then  $\ell(w_\sigma) \leq \ell(v) < \ell(w_\kappa)$ , so  $\vec{\sigma} \neq h(\lambda)$ . Therefore,  $\chi^\lambda(T_{w_\sigma}\tau) = 0$ , by [Theorem 5.1](#), so that  $\chi^\lambda(T_{w_\kappa^-}\tau) = -\chi^\lambda(T_{w_\kappa}\tau)$  as required.  $\square$

More generally, if  $w \in \mathfrak{S}_n$  is a minimal length element of cycle type  $h(\lambda)$  then  $\chi^\lambda(T_w\tau) = \pm\chi^\lambda(T_{w_{h(\lambda)}}\tau)$ .

Using the argument of [Corollary 6.10](#) we can now compute  $\chi^\lambda(T_w\tau)$ , for any  $w \in \mathfrak{S}_n$ . However, if  $\nu$  is a composition of  $n$  then this does not imply that  $\chi^\lambda(T_{w_\nu}\tau) = \pm\chi^\lambda(T_{w_\nu^+}\tau)$  because if  $\ell(w_\nu) > \ell(w_{h(\lambda)})$  then applications of [Lemma 6.9](#) can introduce terms  $\chi^\lambda(T_{w_\sigma}\tau)$  where  $\vec{\sigma} = h(\lambda)$ . See the examples at the end of this section.

We can now prove a stronger version of [Theorem 5.1](#). Let  $\rightsquigarrow$  be the transitive relation on  $\mathfrak{S}_n$  generated by  $w \rightsquigarrow v$  if either  $w \rightarrow v$  or there exists  $s \in S$  such that  $\ell(w) = \ell(sws)$ ,  $v \in \{sw, ws\}$  and  $\ell(v) = \ell(w) - 1$ .

**6.11. Theorem.** *Suppose that  $\lambda = \lambda'$  and  $w \in \mathfrak{S}_n$ . Let  $h(\lambda) = (h_1, \dots, h_{d(\lambda)})$ . Then there exists a polynomial  $a_w^\lambda(a) \in \mathbb{Z}[(q - q^{-1})]$  such that  $\deg_q a_w^\lambda(q) \leq \ell(w) - \ell(w_{h(\lambda)})$  and*

$$\chi^\lambda(T_w\tau) = (-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} a_w^\lambda(q) q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}.$$

Moreover,  $a_w^\lambda(q) \neq 0$  only if there exists a composition  $\kappa$  such that  $w \rightsquigarrow w_\kappa$  and  $\vec{\kappa} = h(\lambda)$ .

*Proof.* As in the proof of [Corollary 5.6](#), by [Theorem 6.5\(a\)](#) and repeated applications of [Lemma 6.9](#) and [Lemma 6.6](#), there exist polynomials  $a_{\sigma w}(q) \in \mathbb{Z}[(q - q^{-1})]$  such that

$$\chi^\lambda(T_w\tau) = \sum_{\sigma} a_{\sigma w}(q) \chi^\lambda(T_{w_\sigma}\tau),$$

where the sum is over compositions  $\sigma$  of  $n$  such that  $w \rightsquigarrow w_\sigma$ . Observe that the different cases of [Lemma 6.9](#) multiply the character values by either  $-1$  or  $q - q^{-1}$ , with  $q - q^{-1}$  appearing only in the case of [Lemma 6.9\(a\)](#) where it arises as the coefficient of  $\chi^\lambda(T_v\tau)$ , where  $v \leq w$  and  $\ell(v) = \ell(w) - 1$ . Therefore,  $\deg_q a_{\sigma w}(q) \leq \ell(w) - \ell(w_\sigma)$ . Set  $a_w^\lambda(q) = \sum_{\kappa} \epsilon_\kappa a_{\kappa w}(q)$ , where the sum is over those compositions  $\kappa$  with  $\vec{\kappa} = h(\lambda)$ . Then

$$\chi^\lambda(T_w\tau) = a_w^\lambda(q) (-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}$$

by [Theorem 5.1](#). All of the claims in the theorem now follow.  $\square$

The polynomials  $a_w^\lambda(q)$  appearing in [Theorem 6.11](#) are, in principle, easy to compute. Examples suggest that if  $w \in C_\mu$  then  $a_w^\lambda(q) \neq 0$  only if  $\mu \geq \lambda$ , where  $\geq$  is the dominance order on  $\mathcal{P}_n$ .

Combining [Proposition 4.12](#) and [Theorem 6.11](#) yields a weak analogue of the Geck-Pfeiffer [Theorem 6.1](#) for the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  indexed by self-conjugate partitions.

**6.12. Corollary.** *Suppose that  $\lambda = \lambda'$  is a self-conjugate partition of  $n$  and  $w \in \mathfrak{A}_w$ . Then*

$$\chi_{\mathcal{A}}^{\lambda\pm}(A_w) = \frac{1}{2} \sum_{C \in \mathcal{C}(\mathfrak{A}_n)} g_{C,w}(q) \chi_{\mathcal{A}}^\lambda(A_{w_C}) \pm \frac{1}{2} (-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} a_w^\lambda(q) q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]},$$

where  $g_{C,w}(q)$  and  $a_w^\lambda(q)$  are the polynomials from [Proposition 6.4](#) and [Theorem 6.11](#), respectively.

*Proof.* By [Proposition 4.12](#) and [Theorem 6.11](#),

$$\chi_{\mathcal{A}}^{\lambda\pm}(A_w) = \frac{1}{2}\chi^\lambda(T_w) \pm \frac{1}{2}(-\sqrt{-1})^{\frac{1}{2}(n-d(\lambda))} a_w^\lambda(q) q^{-\frac{1}{2}n} \prod_{i=1}^{d(\lambda)} \sqrt{[h_i]}.$$

On the other hand, using [Corollary 4.8](#),

$$\chi^\lambda(T_w) = \frac{1}{2}(\chi^\lambda(T_w) + \chi^{\lambda'}(T_w)) = \chi^\lambda(A_w) = \sum_{C \in \mathcal{C}(\mathfrak{A}_n)} g_{C,w}(q) \chi_{\mathcal{A}}^\lambda(A_{w_C}),$$

where the last equality comes from [Proposition 6.4](#). The result follows.  $\square$

Therefore, if  $\chi$  is a character of  $\mathcal{H}_q(\mathfrak{A}_n)$  then the values  $\chi(A_w)$ , for  $w \in \mathfrak{A}_n$ , can be written as a  $\mathcal{Z}$ -linear combination of the character values  $\chi(A_{w_C})$ , where the sum is over the conjugacy classes of  $\mathfrak{A}_n$  and  $w_C \in C_{\min}$ .

[Theorem 6.11](#) also allows us to improve on [Corollary 4.11](#).

**6.13. Corollary.** *Suppose that  $\mathbb{F}$  is a field of characteristic different from 2 that contains square roots*

$$\sqrt{-1} \quad \text{and} \quad \sqrt{[2m+1]} \quad \text{for} \quad 0 \leq m \leq \frac{n-1}{2}.$$

*Then alternating Hecke algebra  $\mathcal{H}_{\mathbb{F},q}(\mathfrak{A}_n)$  is split semisimple.*

*Proof.* By [Proposition 6.4](#) and [Corollary 6.12](#), if  $\chi$  is an irreducible character of  $\mathcal{H}_q(\mathfrak{A}_n)$  then  $\chi(a) \in \mathbb{F}$ , for all  $a \in \mathcal{H}_q(\mathfrak{A}_n)$ . As all of the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  take values in  $\mathbb{F}$  it follows by general nonsense that  $\mathbb{F}$  is a splitting field for  $\mathcal{H}_q(\mathfrak{A}_n)$ . See, for example, [\[2, 7.15\]](#).  $\square$

Finally, by way of example, we show that the obvious generalisations of [Theorem 6.1](#) to the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  fail for both the  $A$ -basis and the  $B$ -basis of  $\mathcal{H}_q(\mathfrak{A}_n)$ . This suggests that [Corollary 6.12](#) (and [Proposition 6.4](#)), may be the best results possible.

**6.14. Example.** We show that the characters values  $\chi(A_w)$  are *not* constant on the minimal length conjugacy class representatives of  $\mathfrak{S}_n$ . Take  $\lambda = (3^3)$  so that  $h(\lambda) = (5, 3, 1)$ . Let  $w = w_{(9)} = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$ . Then  $\chi_{\mathcal{A}}^{\lambda\pm}(A_w) = \frac{1}{2}\chi^\lambda(T_w)$  by [Corollary 4.13](#) and [Theorem 5.1](#). Now consider  $v = s_8 s_5 s_1 s_2 s_3 s_4 s_6 s_7$ . Then  $v$  and  $w$  are conjugate in  $\mathfrak{S}_n$  (although they are not conjugate in  $\mathfrak{A}_n$ ), and they have the same length. By [Corollary 4.13](#),

$$\chi_{\mathcal{A}}^{\lambda\pm}(A_v) = \frac{1}{2}(\chi^\lambda(T_v) \pm \chi^\lambda(T_v \tau)) = \frac{1}{2}(\chi^\lambda(T_w) \pm \chi^\lambda(T_v \tau)),$$

where the last equality comes from [Theorem 6.1](#) since  $v$  and  $w$  are of minimal length in their conjugacy class. On the other hand, using the argument from the proof of [Theorem 6.11](#) shows that

$$\chi^\lambda(T_v \tau) = (q - q^{-1})^2 \chi^\lambda(T_1 T_2 T_3 T_4 T_6 T_7 \tau) = -q^{-4}(q - q^{-1})^2 \sqrt{-1} \sqrt{[3]} \sqrt{[5]}.$$

Consequently,  $\chi_{\mathcal{A}}^{\lambda\pm}(A_v) \neq \chi_{\mathcal{A}}^{\lambda\pm}(A_w)$ .  $\diamond$

**6.15. Example.** Maintaining the notation of the last example with  $\lambda = (3, 3, 3)$ , brute force calculations using code written by the first author in SAGE [\[21\]](#) reveal the following:

$x \in \mathfrak{A}_9$	$\chi_{\mathcal{A}}^{\lambda\pm}(A_x)$	$\chi_{\mathcal{A}}^{\lambda\pm}(B_x)$
$w = s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8$	0	$2^8(q - q^{-1})^8$
$v = s_8 s_5 s_1 s_2 s_3 s_4 s_6 s_7$	$\mp q^{-4} \sqrt{-1} \sqrt{[3]} \sqrt{[5]} (q - q^{-1})^2$	$2^8(q - q^{-1})^8 \mp q^{-4} \sqrt{-1} \sqrt{[3]} \sqrt{[5]} (q - q^{-1})^2$
$u = s_7 s_8 s_5 s_1 s_2 s_3 s_4 s_6$	$\pm q^{-4} \sqrt{-1} \sqrt{[3]} \sqrt{[5]} (q - q^{-1})^2$	$2^8(q - q^{-1})^8 \pm q^{-4} \sqrt{-1} \sqrt{[3]} \sqrt{[5]} (q - q^{-1})^2$

The three permutations  $u$ ,  $v$  and  $w$  are of minimal length in their conjugacy class. All three elements are conjugate in  $\mathfrak{S}_9$  whereas only  $u$  and  $w$  are conjugate in  $\mathfrak{A}_9$ . In particular, this calculation shows that the characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  are not constant on  $A$  or  $B$  basis elements indexed by minimal length conjugacy class representatives. Hence, the obvious generalisation of the Geck-Pfeiffer [Theorem 6.1](#) for the irreducible characters of  $\mathcal{H}_q(\mathfrak{A}_n)$  is not true with respect to either the  $A$  or  $B$  bases of  $\mathcal{H}_q(\mathfrak{A}_n)$ .  $\diamond$

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